Solutions of the Final Exam A

Probability and Statistics

Computer Sciences
23.February.2006

Problem 1

$P(x = 3/\text{peace}) = \dfrac{\exp(-2) * 2^3}{3!}$

$P(x = 3/\text{war}) = \dfrac{\exp(-5) * 5^3}{3!}$

$P(\text{peace}) = 0.8;\ P(\text{war}) = 0.2$

$P(\text{peace}/x = 3) = \dfrac{P(x = 3/\text{peace}) * P(\text{peace})}{P(x = 3)} =$

$\dfrac{P(x = 3/\text{peace}) * P(\text{peace})}{P(x = 3/\text{peace}) * P(\text{peace}) + P(x = 3/\text{war})P(\text{war})} =$

$$\frac{\dfrac{\exp(-2) * 2^3}{3!} * 0.8}{\dfrac{\exp(-2) * 2^3}{3!} * 0.8 + \dfrac{\exp(-5) * 5^3}{3!} * 0.2} = 0.837$$

Problem 2

.

a) $P(x=k) = \frac{k-1}{C_n^2} = \frac{2(k-1)}{n(n-1)}, \quad k>1$

$$\sum_{k=2}^{n} P(x=k) = \sum_{k=2}^{n} \frac{2(k-1)}{n(n-1)} = \frac{2}{n(n-1)} \sum_{k=2}^{n} k = \frac{2}{n(n-1)} \frac{n(n-1)}{2} = 1$$

Example ($n=4$):

k	2	3	4	$\sum$
$P(x=k)$	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{6}{12}$	1

b) $E[X] = \sum_{k=2}^{n} k * P(x=k) = \sum_{k=2}^{n} k * \frac{2(k-1)}{n(n-1)} = \frac{2}{n(n-1)} \sum_{k=2}^{n} k(k-1) =$

$$\frac{2}{n(n-1)} \left[\sum_{k=2}^{n} k^2 - \sum_{k=2}^{n} k \right] = \frac{2}{n(n-1)} \left[\left(\sum_{k=1}^{n} k^2 - 1^2 \right) - \left(\sum_{k=1}^{n} k - 1 \right) \right] =$$

$$\frac{2}{n(n-1)} \left[\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k \right] = \left[\text{using formulas for} \sum_{k=1}^{n} k \text{ and} \sum_{k=1}^{n} k^2 \right] =$$

$$\frac{2}{n(n-1)} \left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right) = \frac{2(n+1)}{3}$$

Problem 3

.

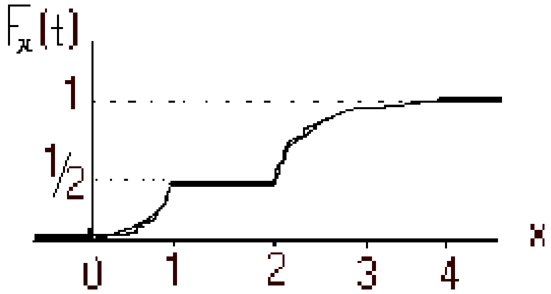
a) Sum of squares of two triangles equal to 1: $A * 1/2 + 2 * B * 1/2 = 1$; or $A + 2B = 1$

Expextation value equal to 5/3: $\int\limits_0^1 Ax^2 dx + \int\limits_2^4 \left(-\frac{B}{2}x^2 + 2Bx\right) dx = 5/3$; or

$\frac{1}{3}Ax^2|_0^1 - \frac{B}{6}x^3|_2^4 + Bx^2|_2^4 = \frac{1}{3}A - \frac{B}{6}(64-8) + B(16-4) = \frac{1}{3}(3A+6B) = \frac{5}{3}$

hence $A = 1$; $B = \frac{1}{2}$

b) $F_x(t) = \{ \begin{array}{ll} 0, & t \leq 0 \\ \int\limits_0^t 1x dx = \dfrac{t^2}{2}, & 0 < t \leq 1 \\ \int\limits_0^1 1x dx = \dfrac{1}{2}, & 1 < t \leq 2 \\ \dfrac{1}{2} + \int\limits_2^4 \left(-\frac{1}{4}x + 1\right) dx = -\dfrac{t^2}{8} + t - 1, & 2 < t \leq 4 \\ 1, & t > 4 \end{array}$

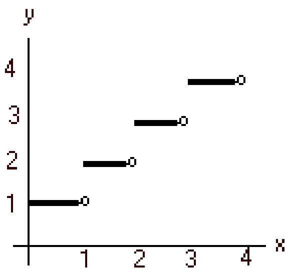


c) $Var[X] = \int\limits_0^1 x^3 dx + \int\limits_2^4 \left(-\frac{1}{4}x^3 + x^2\right) dx - (E[X])^2 = \frac{47}{12} - \frac{25}{9} = \frac{41}{36} = 1.139$

d) $P(0.5 < x < 2/x > 1.5) = \dfrac{P(0.5 < x < 2 \cap x > 1.5)}{1 - P(X < 1.5)} = \dfrac{P(1.5 < x < 2)}{1 - P(X < 1.5)} =$

$= \dfrac{F(x = 2) - F(x = 1.5)}{1 - F(x = 1.5)} = \dfrac{\frac{1}{2} - \frac{1}{2}}{1 - \frac{1}{2}} = 0$

Problem 4



$$P(Y = k) = P(k - 1 \le x < k) = .\int_{k-1}^{k} f_x(x)dx = \lambda \int_{k-1}^{k} e^{-\lambda x}dx =$$

$$= e^{-\lambda(k-1)} - e^{-\lambda k} = e^{-\lambda k}(1 - e^{-\lambda}) = (1 - e^{-\lambda}) * (e^{-\lambda})^{k-1} = p(1 - p)^{k-1},$$

where $p = 1 - e^{-\lambda}$.

Therefore Y has a geometric distribution

Problem 5

.

a) $X_A \sim N(75, 2^2)$; $X_B \sim N(72, 3^2)$;

$Y = \overline{X}_A - \overline{X}_B \sim N(75 - 72; \dfrac{3^2}{6} + \dfrac{2^2}{5}) = N(3; 2.3)$

$P(Y < 0) = \Phi\left(\dfrac{0 - 3}{\sqrt{2.3}}\right) = \Phi(-1.978) = 1 - 0.974 = 0.026$

b) $\epsilon = \dfrac{60-58}{2} = 1$

$1-\alpha = 0.90;\ \alpha = 0.1;\ \Phi\left(z_{\dfrac{\alpha}{2}}\right) = 1-\dfrac{\alpha}{2} = 0.95;\ \ z_{\dfrac{\alpha}{2}} = 1.645$

$n \geq \left(\frac{z_{\frac{\alpha}{2}}\sigma}{\epsilon}\right)^2 = \left(\dfrac{1.645*6}{1}\right)^2 = 97.417;\ \ n = 98$

Problem 6

.

$\overline{X} = \dfrac{1}{25} * 65 = 2.6;\ \widehat{S}^2 = \dfrac{1}{24} * 16 = 0.6666;\ \widehat{S} = 0.816\,;$

a) $$\overline{X} - t_{0.01}(24)\frac{\widehat{S}}{\sqrt{n}} < \mu < \overline{X} + t_{0.01}(24)\frac{\widehat{S}}{\sqrt{n}};$$

$2.6 - 2.492 * \frac{0.816}{\sqrt{25}} < \mu < 2.6 + 2.492 * \frac{0.816}{\sqrt{25}}$

$2.1933 < \mu < 3.0067$

b)
$H_0 : \mu = 2$
$H_1 : \mu > 2$

$T = \dfrac{\overline{X}-2}{\widehat{S}/\sqrt{25}} = \dfrac{2.6-2}{0.816/\sqrt{25}} = 3.6765 > t_{0.01}(24) = 2.492$

or (another way)

critical value $C_2 = \mu + t_{0.01}(24) * \widehat{S}/\sqrt{n} = 2 + 2.492 * 0.816/\sqrt{25}$
$= 2.4067 < \overline{X} = 2.6$

We shoul reject H_0 (that no difference between a mean value of a sample and expectation value).

This medicine influences to a driver reaction.