Solutions to Laplace Transform Exercise

(1) (a) Since $\cos^2 3t = \dfrac{1+\cos 6t}{2}$, $\mathfrak{L}\{e^{-2t}\cos^2 3t - 3t^2e^{3t}\}$

$= \mathfrak{L}\left\{e^{-2t}\left(\frac{1}{2}+\frac{1}{2}\cos 6t\right) - 3e^{3t}t^2\right\} = \dfrac{1}{2(s+2)} + \frac{1}{2}\dfrac{s+2}{(s+2)^2+36} - \dfrac{6}{(s-3)^3}$

$= \dfrac{1}{2(s+2)} + \frac{1}{2}\dfrac{s+2}{s^2+4s+40} - \dfrac{6}{(s-3)^3}.$

(b) $\because \mathfrak{L}\{e^{-t}t^{100}\} = \dfrac{100!}{(s+1)^{101}}$, $\therefore \mathfrak{L}\left\{\dfrac{d^{100}}{dt^{100}}(e^{-t}t^{100})\right\} = s^{100}\times\dfrac{100!}{(s+1)^{101}}$

because $\dfrac{d^k}{dt^k}(e^{-t}t^{100}) = 0$ when $t=0$ whenever $0 \le k \le 99$. Therefore,

$\mathfrak{L}\left\{e^t\cdot\dfrac{d^{100}}{dt^{100}}(e^{-t}t^{100})\right\} = 100!\times\dfrac{(s-1)^{100}}{s^{101}}.$

(2) (a) Decomposing $F(s)$ into partial fractions, we obtain

$\dfrac{4s+5}{(s-2)^3(s+3)} = \dfrac{7}{125\,(s+3)} - \dfrac{7}{125\,(s-2)} + \dfrac{7}{25\,(s-2)^2} + \dfrac{13}{5\,(s-2)^3}.$

Therefore, $\mathfrak{L}^{-1}\{F(s)\} = \dfrac{7}{125}e^{-3t} - \dfrac{7}{125}e^{2t} + \dfrac{7}{25}te^{2t} + \dfrac{13}{10}t^2e^{2t}.$

(b) $\mathfrak{L}^{-1}\left\{\dfrac{1}{s^2(s^2+3s-4)}\right\} = \mathfrak{L}^{-1}\left\{\dfrac{1}{5\,(s-1)} - \dfrac{3}{16s} - \dfrac{1}{80\,(s+4)} - \dfrac{1}{4s^2}\right\}$

$= \dfrac{1}{5}e^t - \dfrac{3}{16} - \dfrac{1}{80}e^{-4t} - \dfrac{1}{4}t.$

(c) We decompose $F(s)$ into partial fractions to obtain

$$\frac{s}{(s-1)^2(s^2+2s+5)} = \frac{1}{16\,(s-1)} + \frac{1}{8\,(s-1)^2} - \frac{1}{16}\frac{s}{s^2+2s+5} - \frac{5}{16}\frac{1}{s^2+2s+5}.$$

Since $s^2+2s+5 = (s+1)^2+2^2$, we conclude that

$$\mathfrak{L}^{-1}\{F(s)\} = \frac{e^t}{16} + \frac{te^t}{8} - \frac{e^{-t}}{16}(\cos 2t - \frac{\sin 2t}{2}) - \frac{5}{32}e^{-t}\sin 2t.$$

(3) (a) We transform the differential equation to obtain

$$(s^2+k^2)Y(s) = \frac{2\omega}{s^2+\omega^2},$$

from which it follows that $Y(s) = \dfrac{2\omega}{(s^2+\omega^2)(s^2+k^2)}$. There are *two cases* to consider:

(i) If $\omega = k$, then $y(t) = \mathfrak{L}^{-1}\left\{\dfrac{2\omega}{(s^2+\omega^2)^2}\right\} = \dfrac{\sin\omega t - \omega t\cos\omega t}{\omega^2}$.

(ii) If $\omega \neq k$, then $y(t) = \dfrac{2\omega}{k^2-\omega^2}\left(\dfrac{\sin\omega t}{\omega} - \dfrac{\sin kt}{k}\right)$.

(b) We apply Laplace Transform to the equation to obtain

$$(s^2+4s+5)Y(s) + s + 4 = \frac{100}{s+2}.$$

Hence $Y(s) = -\dfrac{s}{s^2+4s+5} - \dfrac{4}{s^2+4s+5} + \dfrac{100}{(s+2)(s^2+4s+5)}$. Using the fact that $\dfrac{100}{(s+2)(s^2+4s+5)} = \dfrac{100}{s+2} - \dfrac{100s}{s^2+4s+5} - \dfrac{200}{s^2+4s+5}$, one obtains

$$\begin{aligned} y(t) &= \mathfrak{L}^{-1}\left\{-\frac{s}{s^2+4s+5} - \frac{4}{s^2+4s+5} + \frac{100}{(s+2)(s^2+4s+5)}\right\} \\ &= \mathfrak{L}^{-1}\left\{-\frac{s}{(s+2)^2+1} - \frac{4}{(s+2)^2+1} + \frac{100}{(s+2)[(s+2)^2+1]}\right\} \\ &= -100e^{-2t} - 101e^{-2t}(\cos t - 2\sin t) - 204e^{-2t}\sin t. \end{aligned}$$

(c) We transform the system of differential equations to obtain the following system of linear algebraic equations:

$$\begin{cases} (s-1)X(s) - Y(s) &= \dfrac{1}{s^2} \\ -4X(s) + (s-1)\,Y(s) &= 2 \end{cases}.$$

Cramer's rule then gives $X(s) = \dfrac{2s^2+s-1}{s^2(s-3)(s+1)} = \dfrac{1}{3s^2} - \dfrac{5}{9s} + \dfrac{5}{9\,(s-3)}$ and $Y(s) = \dfrac{2s^3-2s^2+4}{s^2(s-3)(s+1)} = \dfrac{8}{9s} - \dfrac{4}{3s^2} + \dfrac{10}{9\,(s-3)}$. Taking inverse transform, we obtain

$$x(t) = \frac{5}{9}e^{3t} + \frac{1}{3}t - \frac{5}{9} \quad \text{and} \quad y(t) = \frac{10}{9}e^{3t} + \frac{8}{9} - \frac{4}{3}t.$$

(d) Laplace Transformation, when applied to the equation, gives $(s^2-6s+9)Y(s) + s - 8 = \dfrac{2}{s-3}$, or

$$\begin{aligned} Y(s) &= -\frac{s}{(s-3)^2} + \frac{8}{(s-3)^2} + \frac{2}{(s-3)^3} \\ &= -\frac{1}{s-3} + \frac{5}{(s-3)^2} + \frac{2}{(s-3)^3}. \end{aligned}$$