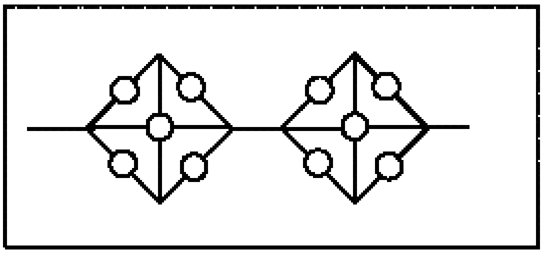
Solutions of the Midterm Exam

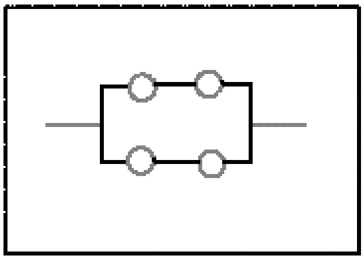
Management of Technology

3.May.2005

1 Problem 1



ציור ב'



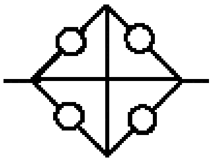
ציור א'

We can see, that the probability to work for an electric circuit A is $p^2 + p^2 - p^2 p^2$ or $2p^2 - p^4 = \frac{17}{81}$. Substituting $x = p^2$, we have $81x^2 - 162x + 17 = 0$. Hence $x_1 = \frac{1}{9}$ or $x_2 = \frac{17}{9}$. Probability p should not be greater then one, hence x_2 should not be taken into account. From $x = \frac{1}{9}$ we have $p_1 = \frac{1}{3}$ or $p_2 = -\frac{1}{3}$. Because $p \geq 0$, the only solution is $p = \frac{1}{3}$.

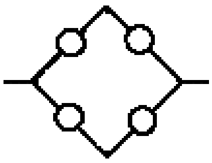
An electric circuit B can be considered as sequence of two similar circuits B_1 with a probability of q each. Therefore the total probability of a circuit B will be equal to q^2.

To order to find a probability q we consider two situations:

a) the central block of B_1 is working (with a probability of p) . Obvious, that a probability, that a whole circuit B_1 is working, is equal to 1.



b) the central block is out of order (with a probability of $1-p$) . Then a circuit B_1 is similar to a circuit A, and its probability to work is equal to $\frac{17}{81}$.



Therefore, a total probability, that a circuit B_1 is working, will be equal to

$$q = p * 1 + (1-p) * \frac{17}{81} = \frac{1}{3} + \frac{2}{3}\frac{17}{81} = \frac{115}{243} = 0.473,$$

and a total probability, that a whole electric circuit B is working, will be equal to

$$0.473^2 = 0.224.$$

2 Problem 2

Let us denote p_A, p_B, and p_C - probabilities to order a production from factories A, B, and C, correspondently. Then $p_A + p_B + p_C = 1$, and substituting $x = p_B = p_C$ and taking to account that $p_A = 3x$, we have $3x + x + x = 1$, hence

$$x = p_B = p_C = 0.2, p_A = 0.6.$$

The total probability to draw randomly a defective ball will be equal to

$$p = 0.6 * 0.03 + 0.2 * 0.02 + 0.2 * 0.04 = 0.03,$$

therefore

a) the probability to draw randomly a good ball will be equal to

$$1 - p = 0.97$$

b) Using Bayes theorem we have

$$p(\text{good}/B) = \frac{p(B/\text{good})p(B)}{p(\text{good})} = \frac{(1-0.02)*0.2}{1-0.03} = 0.202$$

c) This is a binomial distribution

$$P(x = k) = C_3^k p^k (1-p)^{n-k},$$

where k- number of good balls from drawn 3 balls, p- probability to draw a defective ball, hence

$$P(x = 3) = 1 * 0.03^0 * 0.97^3 = 0.912\,67$$
$$P(x = 2) = 3 * 0.03^1 * 0.97^2 = 0.08\,468$$
$$P(x = 1) = 3 * 0.03^2 * 0.97^1 = 0.002\,62$$
$$P(x = 0) = 1 * 0.03^3 * 0.97^0 = 0.000027$$

or

k	3	2	1	0	$\sum$
$P(x = k)$	0.912 67	0.08 468	0.002 62	0.000027	1

3 Problem 3

A patient should make a Rentgen photo:

a) exactly four times: when first three times - failure, and at the forth time - success:

$$(1-p)^3 p = 0.4^3 * 0.6 = 0.0384;$$

b) more then two times: when first two times - failure:

$$(1-p)^2 = 0.4^2 = 0.16$$

c) $P(x=3|\text{failure in first}) = \frac{P(x=3 \bigcap \text{failure in first})}{P(\text{failure in first})} = \frac{P(x=3)}{1-p}$
$= \frac{(1-p)^2 p}{1-p} = (1-p)p = 0.4 * 0.6 = 0.24$
[another approach]: According to the property of "no memory",
$P(x=3|\text{failure in first}) = P(x=2) = (1-p)p = 0.4 * 0.6 = 0.24$

d) Expectation

$$E[X] = \frac{1}{0.6} = 1.667$$

4 Problem 4

a)

$$P(x=n, y=k) = C_n^k p^k (1-p)^{n-k} * P(x=n)$$

b)

$\frac{Y(\text{number of defective lamps})}{X(\text{number of lamps})}$	0	1	2	3	$P(X)$
1	$\frac{9}{30}$	$\frac{1}{30}$	0	0	$\frac{1}{3}$
2	$\frac{81}{300}$	$\frac{18}{300}$	$\frac{1}{300}$	0	$\frac{1}{3}$
3	$\frac{729}{3000}$	$\frac{243}{3000}$	$\frac{27}{3000}$	$\frac{1}{3000}$	$\frac{1}{3}$
$P(Y)$	$\frac{2439}{3000}$	$\frac{523}{3000}$	$\frac{37}{3000}$	$\frac{1}{3000}$	1

c)

$$E[X] = \frac{1+2+3}{3} = 2;$$
$$Var[X] = \frac{1+4+9}{3} - E[X]^2 = 0.667$$

$$E[Y] = \frac{0*2439 + 1*523 + 2*37 + 3*1}{3000} = \frac{600}{3000} = 0.2$$

$$Var[Y] = \frac{0*2439 + 1*523 + 4*37 + 9*1}{3000} - 0.2^2 = 0.186\,7$$

$$E[XY] = \frac{1}{30}(0*1*9 + 1*1*1)$$

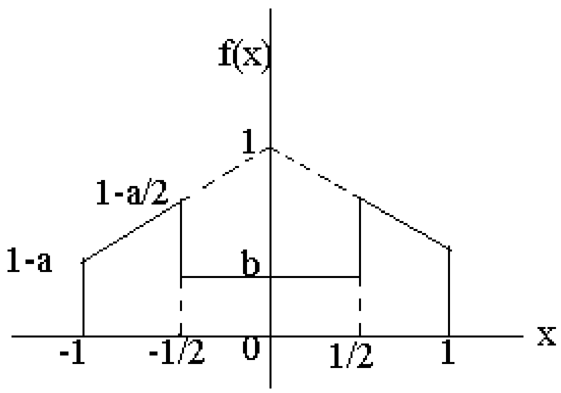
$$+\frac{1}{300}(0*2*81+1*2*18+2*2*1)$$
$$+\frac{1}{3000}(0*3*729+1*3*243+2*3*27+3*3*1) =$$
$$= \frac{100+400+900}{3000} = 0.4667$$

$$\mathrm{cov}(X,Y) = E[XY] - E[X]E[Y] = 0.4667 - 2*0.2 = 0.0667$$

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{Var[X]Var[Y]}} = \frac{0.0667}{\sqrt{0.667*0.1867}} = 0.189$$

5 Problem 5

$$f(x) = \begin{cases} 0, & x < -1 \\ 1+ax, & -1 \le x < -1/2 \\ b, & -1/2 \le x < 1/2 \\ 1-ax, & 1/2 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$



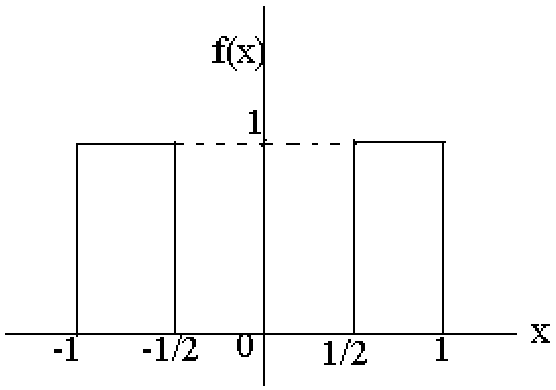
a) Function $f(x)$ is simmetrical, therefore $E[X] = 0$

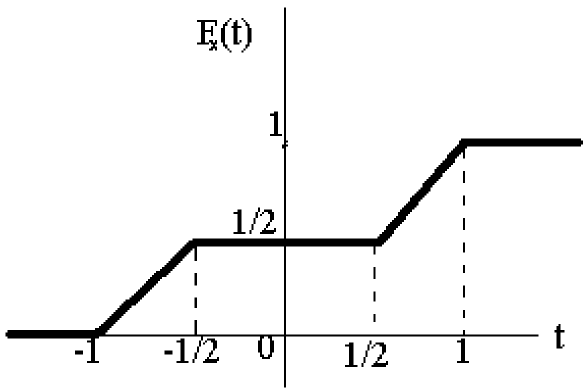
b) Pdf (Probability density function) $f(x) \ge 0$, hence $b \ge 0$. When $x = -1, f(x) = 1 - a \ge 0$. Hence $a \le 1$

Area under pdf is equal to 1. Hence $\frac{1}{2}((1-a) + (1-\frac{a}{2})) * 1 + b * 1 = 1$; $1 - \frac{3}{4}a + b = 1$; $b = \frac{3}{4}a$. As $b \ge 0$, then $a > 0$. As $a \le 1$, then $b \le \frac{3}{4}$.

Finally, $[0 \le a \le 1]$; $[0 \le b \le \frac{3}{4}]$ and $b = \frac{3}{4}a$

c) Mimimal value for b is 0, then a also will be equal to 0

d) 

e) 

f) $Var[X] = \int_{-1}^{-1/2} x^2 dx + \int_{1/2}^{1} x^2 dx - 0^2 = \frac{1}{3}(1 - \frac{1}{8}) + \frac{1}{3}(1 - \frac{1}{8}) = \frac{14}{24} = 0.583$

g) $P(\frac{1}{4} \leq x \leq \frac{3}{4}) = F(\frac{3}{4}) - F(\frac{1}{4}) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

6 Problem 6

$X\tilde{\ }N(100, 400)$; $\mu = 100$; $\sigma = 20$;

a) $P(x < 100) = P(x < \mu) = 0.5$

b) $P(x > 112) = 1 - \Phi\left(\frac{112-100}{20}\right) = 1 - \Phi(0.6) = 1 - (0.2257 + 0.5) = 0.2743$

c) $P(100 < x < 140) = \Phi\left(\frac{140-100}{20}\right) - \Phi\left(\frac{100-100}{20}\right) = \Phi(2.0) - \Phi(0) = (0.4772 + 0.5) - 0.5 = 0.4772$

d) We should find x_0 such that $P(x > x_0) = 0.1$ or $P(x < x_0) = 0.9$. It will correspond to the value

$$z_0 = \frac{x_0 - \mu}{\sigma}$$

where $\Phi(z_0) = 0.9$. In our table we are looking for $0.9 - 0.5 = 0.4$ in the body of a table. The nearest value in the table is $z_0 = 1.28$, because $\Phi(1.28) = 0.3997$. Hence $x_0 = \sigma z_0 + \mu = 20 * 1.28 + 100 = 125.6$. The minimal IQ is 126.