Problem 1

$P_L = 0.9;\ P_T = 0.85;\ P(L) = 0.4; P(T) = 0.6$

a) $P_a = P(L) * P_L + P(T) * (1 - P_T) = 0.4 * 0.9 + 0.6 * 0.15 = 0.45$

b)$P_b = \dfrac{P(L) * P_L}{P_a} = \dfrac{0.4 * 0.9}{0.45} = 0.8$

c)$P_c = \dfrac{P(L) * P_L * P_L}{P(L) * P_L * P_L + P(T) * (1 - P_T) * (1 - P_T)} =$

$= \dfrac{0.4 * 0.9 * 0.9}{0.4 * 0.9 * 0.9 + 0.6 * 0.15 * 0.15} = 0.96$

Problem 2

a) $P(X = 0, Y = 0) = \dfrac{C_3^2}{C_8^2} = \dfrac{3}{28};\ P(X = 0, Y = 1) = \dfrac{C_3^1 C_3^1}{C_8^2} = \dfrac{9}{28};$ etc...

$X\backslash Y$	0	1	2	$P(X)$
0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
1	$\frac{6}{28}$	$\frac{6}{28}$		$\frac{12}{28}$
2	$\frac{1}{28}$			$\frac{1}{28}$
$P(Y)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	1

b) dependent because, for example,

$$P(X = 0, Y = 0) = \frac{3}{28} \neq P(X = 0) * P(Y = 0) = \frac{15}{28} * \frac{10}{28}$$

$$E[X] = 1 * \frac{12}{28} + 2 * \frac{1}{28} = 0.5$$

$$E[Y] = 1 * \frac{15}{28} + 2 * \frac{3}{28} = 0.75$$

$$E[XY] = 1 * 1 * \frac{6}{28} = 0.214$$

$$Cov(X,Y) = E[XY] - E[X] * E[Y] = 0.214 - 0.5 * 0.75 = -0.161$$

correlated because $Cov(X,Y) \neq 0$

c) $P(X = Y) = P(X = 0, Y = 0) + P(X = 1, Y = 1) = \frac{3}{28} + \frac{6}{28} = \frac{9}{28}$

Problem 3

a) $\int_{-\infty}^{\infty} f_X(x)dx = 1$, hence $\int_{-1}^{0} ax^2dx + \int_{0}^{1} bxdx = 1$,

or $\frac{a}{3}x^3|_{-1}^{0} + \frac{b}{2}x^2|_{0}^{1} = 1$, or $\frac{a}{3} + \frac{b}{2} = 1$, or $2a + 3b - 6 = 0$

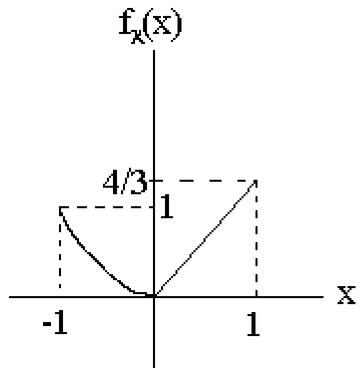
Expextation value $E[X] = \int_{-\infty}^{\infty} x\, f_X(x)dx = \frac{7}{36}$, therefore

$\int_{-1}^{0} ax^3dx + \int_{0}^{1} bx^2dx = \frac{7}{36}$, or $\frac{a}{4}x^4|_{-1}^{0} + \frac{b}{3}x^3|_{0}^{1} = \frac{7}{36}$,

or $-\frac{a}{4} + \frac{b}{3} = \frac{7}{36}$, or $9a - 12b + 7 = 0$

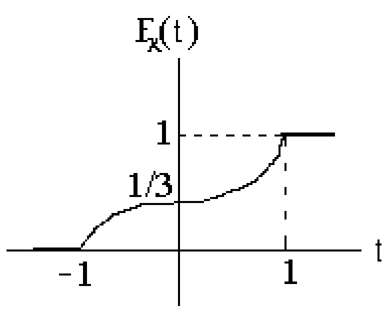
$$\begin{cases} 2a + 3b - 6 = 0 \\ 9a - 12b + 7 = 0 \end{cases}$$

$a = 1;\ b = \frac{4}{3}$



b) $F_x(t) = \begin{cases} 0, & x < -1 \\ \int_{-1}^{t} x^2 dx, & -1 \le x < 0 \\ \int_{-1}^{0} x^2 dx + \int_{0}^{t} \frac{4}{3} x dx, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$; hence

$$F_x(t) = \begin{cases} 0, & x < -1 \\ \frac{1}{3}(t^3 + 1), & -1 \le x < 0 \\ \frac{1}{3} + \frac{2}{3}t^2, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$



c) $Var[X] = E[X^2] - (E[X])^2 = \int_{-\infty}^{\infty} x^2 f_X(x)dx - \left(\frac{7}{36}\right)^2 =$

$= \int_{-1}^{0} x^4 dx + \int_{0}^{1} \frac{4}{3}x^3 dx - \left(\frac{7}{36}\right)^2 = \frac{1}{5} + \frac{4}{3*4} - \frac{49}{1296} = 0.495$

d) $P(-0.5 < x \le 0.5) = F_x(0.5) - F_x(-0.5) =$

$=\left(\frac{1}{3} + \frac{2}{3}t^2\right)|_{t=0.5} - \frac{1}{3}(t^3 + 1)|_{t=-0.5} = 0.208$

Problem 4

$X \sim U(3,4),\ \mu_x = 3.5$ and $\sigma_x^2 = \dfrac{(4-3)^2}{12} = \dfrac{1}{12}$

Denote $S = \sum_{i=1}^{1000} X_i$. - time for 1000 bottles.

$S \sim N(1000 * \mu_x, 1000 * \sigma_x^2) = N(3500, 83.33)$

a) 10 years=3652 days, and $P(S \ge 3652) = 1 - P(S < 3652) =$

$1 - \Phi\left(\frac{3652-3500}{\sqrt{83.33}}\right) \approx 0$

b) For how much time this store is enough? (Probability 95%)
This question is not correct. You should ask either

b1) What is a minimum time, that you sure 95% the store is enough?
Then you apply one-tail solution
$\Phi\left(\frac{z-3500}{\sqrt{83.33}}\right) = 0.95;\ \frac{z-3500}{\sqrt{83.33}} = 1.63;$

$z = 3500 - 1.645 * \sqrt{83.33} = 3485.0$ (days)
or
b2) What is the range (in days), including 95% of this distribution?
$\frac{z-3500}{\sqrt{83.33}} = 1.96;$
$z_{\min} = 3500 - 1.96 * \sqrt{83.33} = 3482$
$z_{\max} = 3500 + 1.96 * \sqrt{83.33} = 3518$

Problem 5

$\mu = 100;\ \sigma = 1.75;\ n = 10$
$X = \{102, 97, 101, 103, 101, 98, 99, 104, 103, 98\}$

$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i =$
$= \frac{1}{10}(102 + 97 + 101 + 103 + 101 + 98 + 99 + 104 + 103 + 98) = 100.6$

$\widehat{S}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \frac{1}{9} * 54.4 = 6.0444;\ \widehat{S} = \sqrt{6.0444} = 2.4585$

a1) Test mean value with $\alpha = 0.05;\ c = 100.0$

$$\begin{matrix} H_0 : \mu = c \\ H_A : \mu \neq c \end{matrix}\ ;\ t_\alpha = t_{0.025}(9) = 2.26$$

$$T = \frac{\overline{X} - c}{\widehat{S}/\sqrt{n}} = \frac{100.6 - 100.0}{2.4585/\sqrt{10}} = 0.772$$

$T < t_\alpha$; We can not reject H_0. No significant difference for mean value

a2) Test mean value with $\alpha = 0.01$
$t_\alpha = t_{0.005}(9) = 3.25$
$T < t_\alpha$; We can not reject H_0. No significant difference for mean value

b1) Test varibility with $\alpha = 0.05;\ c = 1.75^2 = 3.0625$

Manufacturing is good if variability is small, therefore we should check the hypothesis $\sigma^2 > c$

$$\begin{matrix} H_0 : \sigma^2 = c \\ H_A : \sigma^2 > c \end{matrix}\ ;\ \chi^2_\alpha = \chi^2_{0.05}(9) = 16.92$$

$$\chi^2 = \frac{(n-1)\widehat{S}^2}{c} = \frac{9 * 6.0444}{3.0625} = 17.763$$

$\chi^2 > \chi^2_\alpha$ We can reject H_0 with a level of significance 0.05. Variability of bolts is greater then permitted.

b2) Test varibility with $\alpha = 0.01$;

$$\begin{matrix} H_0 : \sigma^2 = c \\ H_A : \sigma^2 > c \end{matrix}\ ;\ \chi^2_\alpha = \chi^2_{0.01}(9) = 21.67$$

$\chi^2 < \chi^2_\alpha$ We can not reject H_0 with a level of significance 0.01. No significant difference for variability

Problem 6

$n_1 = 200;\ n_2 = 100;\ p_1 = 0.55;\ p_2 = 0.44;\ \alpha = 0.05$

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{200 * 0.55 + 100 * 0.44}{200 + 100} = 0.513;\ \ q = 1 - p = 0.487$$

$$Z = \frac{p_1 - p_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.55 - 0.44}{\sqrt{0.513 * 0.487(\frac{1}{200} + \frac{1}{100})}} = 1.797$$

a) $\begin{matrix} H_0 : p_1 = p_2 \\ H_A : p_1 \neq p_2 \end{matrix}$; $z_\alpha = z_{0.025} = 1.96$

$Z < z_\alpha$; We can not reject H_0. No significant difference between districts.

b) $\begin{matrix} H_0 : p_1 = p_2 \\ H_A : p_1 > p_2 \end{matrix}$; $z_\alpha = z_{0.05} = 1.645$

$Z > z_\alpha$; We can reject H_0. The candidate is preferred in district A.