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Finite element analysis of propagation modes in a waveguide: Effect of gravitational field

Arti Vaish* and Harish Parthasarathy

Division of Electronics and Communication Engineering,
Netaji Subhas Institute of Technology, Dwarka Sector 3, New Delhi 110075, India
** Corresponding author: vaisharti@rediffmail.com*

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Abstract

The law of Electromagnetic wave propagation inside a waveguide is governed by the Helmholtz equation. This equation is derived from the standard wave equation by assuming sinusoidal time dependence. Such an equation is valid if the space-time manifold is flat, i.e., Minkowskian. In the presence of a gravitational field, according to Einstein's general theory of relativity, the space-time manifold becomes curved and the geometry of such a curved manifold is described by a Riemannian metric. Consequently, the wave equation in such a curved space-time needs to be modified to account for the curvature. In addition, the assumption of sinusoidal time dependence gives a modified Helmholtz equation. Since the metric in a curved space-time manifold can be expressed in terms of the gravitational potential, the gravitational potential will enter into the curved space-time wave equation. Such an equation can be derived from a variational principle. In this paper, this variational problem has been solved using a finite element method on rectangular waveguide. The shifts in the modes of propagation induced by the gravitational potential are obtained and compared by considering a numerical example.

Keywords: General theory of relativity, electromagnetism, metric tensor, generalized Laplace operator, test functions, finite element method.

1 Introduction

The scalar wave equation in flat (or rectangular) space-time is the standard wave equation [1]. When the dependence on time and space (z-coordinate) is sinusoidal and exponential, respectively, this wave equation reduces to the two-dimensional Helmholtz equation. It can be solved numerically by using the finite element method for a waveguide (of rectangular cross-section) with specified potential on the boundary [2]. When the metric of space-time is curved, the scalar wave equation considered above is no longer coordinate covariant and one must use the Laplace-Beltrami operator, i.e., wave operator in curvilinear coordinates. Such a technique is particularly appropriate for describing the propagation of electromagnetic waves in a gravitational field. It must, however, be remembered that the propagation of electromagnetic waves in a curved space-time is not described by the scalar wave equation [3, 4] and one must describe the dynamics of the four vector potential. However, this exercise serves as a starting point for investigating numerical methods for wave propagation in a gravitational field. In addition to the gravitation and Electromagnetic fields, the present work is also related to the general theory of relativity and it is appropriate to discuss it in brief.

2 General theory of relativity: Background

In the present work, the concept of relativity given by Einstein in his preliminary predictions has been applied. Einstein's preliminary prediction explains how a ray of light from a distant star, passing near the Sun, would appear to be attracted, or bent slightly, in the direction of the Sun's mass. At the same time, light radiated from the Sun would interact with the Sun's mass, resulting in a slight change toward the infrared end of the Sun's optical spectrum [5].

About 1912, Einstein began a new phase of his gravitational research, with the help of his mathematician friend Marcel Grossmann, by phrasing his work in terms of the tensor calculus of Tullio Levi-Civita and Gregorio Ricci-Curbastro. The tensor calculus greatly facilitated calculations in four-dimensional space-time, a notion that Einstein had obtained from Hermann Minkowski's mathematical elaboration of Einstein's own special theory of relativity [1]. Einstein named this new work as the *general theory of relativity.* After a number of false starts, he published the definitive form of the general theory in late 1915. In this, the gravitational field equations were covariant, i.e., similar to Maxwell's equations. The field equations have the similar form in all equivalent frames of reference. To their advantage from the beginning, the covariant field equations gave the observed perihelion motion of the planet Mercury. Over the past 60 years, the original form of Einstein's theory of general relativity has been verified numerous times, especially during solar-eclipse expeditions when Einstein's light-deflection prediction could be tested. In 1915, Einstein developed the theory of general relativity in conjunction with the objects accelerated with respect to one another. This theory explains apparent conflicts between the laws of relativity and gravity. To resolve these conflicts, he developed an entirely new approach to the concept of gravity, which is based on the principle of equivalence. The principle of equivalence holds that forces produced by gravity are in every way equivalent to forces produced by acceleration, and so it is theoretically impossible to distinguish between gravitational and accelerational forces by experiment.

Thus, Newton's hypothesis, i.e., every object attracts every other object in direct proportion to its mass, is replaced by the relativistic hypothesis, i.e., the continuum is curved in the neighborhood of massive objects. Einstein's law of gravity simply states that the world line of every object is a *geodesic* in the continuum. A *geodesic* is the shortest distance between two points, but in curved space it is not generally a straight line. Similarly, *geodesics* on the surface of the earth are great circles, which are not straight lines on any ordinary map.

Since 1915, the theory of relativity has been developed extensively among others by Einstein and by the British astronomers James Hopwood Jeans, Arthur Stanley Eddington, and Edward Arthur Milne, by the Dutch astronomer Willem de Sitter, and by the German-American mathematician Hermann Weyl [6, 7, 8]Most of their efforts are to extend the theory of relativity under electromagnetic phenomena. Although some progress has been made in this area, these efforts have been marked thus far by less success. No complete development of this application of the theory has yet been generally accepted.

In 1928, a relativistic electron theory was developed by the British mathematician and physicist Paul Dirac, and subsequently a satisfactory quantized field theory, called *quantum electrodynamics*, was evolved. It unifies the concepts of relativity and quantum theory in relation to the interaction between electrons, positrons, and electromagnetic radiation [9].

In recent years, Hawking [10] made an attempt of full integration of quantum mechanics with relativity theory. Many attempts have been made in this work, yet very few people have studied the effect of gravitational field on the electromagnetic, i.e., the frequency of propagation of waveguide. Thus, the present work aims to investigate the effects of gravitational field on the frequency of propagation modes of the waveguide using the finite element method.

The covariant field equation arising in the field of general theory of relativity is the scalar wave equation, which describes the propagation of scalar wave field in the presence of gravitation, and is described below.

3 Problem formulation

The action for the scalar field V is given by

$$S[V] = \int g^{\mu\nu} V_{,\mu} V_{,\nu} \sqrt{g} d^4x \tag{1}$$

where

$$V_{,\eta} = \frac{\partial V}{\partial X^{\eta}} \qquad \eta = \mu, \nu$$

and $g^{\mu\nu}$ and g and are the contravariant metric tensor and the determinant of covariant metric tensor, respectively.

It is known from the Jacobian theory that $S[V]$ is invariant under diffeomorphisms of the space-time manifold.

The action principle

$$\delta S[V] = 0 \tag{2}$$

gives the scalar wave equation

$$(g^{\mu\nu} \sqrt{g} V_{,\nu})_{,\mu} = 0 \tag{3}$$

Assume that the integral is carried out over the region D. The coordinates (x^0, x^1, x^2, x^3) are (ict, x, y, z)

and for the Newtonian metric

$$ds^2 = dx^2 + dy^2 + dz^2 - C^2 \left(1 + \frac{2U}{C^2}\right) dt^2 \tag{4}$$

we have

$$g_{00} = \left(1 + \frac{2U}{C^2}\right), g_{11} = g_{22} = g_{33} = 1$$

other $g'_{ij}s$ being zero. We then get

$$g^{\mu\nu} V_{,\mu} V_{,\nu} \sqrt{g} = -\frac{1}{C^2} \left(1 + \frac{2U}{C^2}\right)^{-1/2} V_{,t}^2 + \left(1 + \frac{2U}{C^2}\right)^{1/2} |\nabla V|^2 \tag{5}$$

where $V_{,t} = \frac{\partial V}{\partial t}$. For sinusoidal time dependence, the action is, therefore, given by

$$S[V] = \int \left[\left(1 + \frac{2U}{C^2}\right)^{1/2} |\nabla V|^2 - k^2 \left(1 + \frac{2U}{C^2}\right)^{-1/2} V^2 \right] dxdydz \qquad (6)$$

where k is the wave number. Setting the variation $\delta S[V] = 0$ then gives

$$\left(\nabla, \left(1 + \frac{2U}{C^2}\right)^{1/2} \nabla V\right) + k^2 \left(1 + \frac{2U}{C^2}\right)^{-1/2} V = 0 \qquad (7)$$

We define the function

$$\phi(x, y, z) = \left(1 + \frac{2U}{C^2}\right)^{1/2} \qquad (8)$$

and then the modified Helmholtz equation becomes

$$(\nabla, \phi \nabla V) + k^2 \phi^{-1} V = 0 \qquad (9)$$

Eq. (9) is a generalized eigenvalue problem. The approximate minimization of the action function can be carried out using the finite element method. In the finite element method, the volume V is divided into pixels and within each pixel, we approximate the potential by a linear function. In this work, the two-dimensional problem is considered as it is easier to analyze. For the two-dimensional problem, metric tensor is given by

$$ds^2 = -\left(1 + \frac{2U}{C^2}\right) C^2 dt^2 + dx^2 + dy^2 \qquad (10)$$

so that

$$\begin{aligned}
x^0 &= ict && (11a)\\
x^1 &= x && (11b)\\
x^2 &= y && (11c)\\
g_{00} &= \left(1 + \frac{2U}{C^2}\right) && (11d)\\
g_{11} &= g_{22} = 1 && (11e)\\
g &= \left(1 + \frac{2U}{C^2}\right) && (11f)\\
& && (11g)
\end{aligned}$$

Then, the action for the modified two-dimensional Helmholtz equation is

$$S[V] = \int [(\left(1 + \frac{2U}{C^2}\right)^{1/2} |\nabla V|^2 - k^2 \left(1 + \frac{2U}{C^2}\right)^{-1/2} V^2]dxdy. \quad (12)$$

Assume that the wave guide is rectangular and the gravitational potential U is generated by a piece of point matter located at the point $(R, 0)$. Then,

$$U(x, y) = -\frac{GM}{((x-R)^2 + y^2)^{1/2}} \quad (13)$$

where G is the gravitational constant and M is mass of the earth. The waveguide cross section is $[0, A] \times [0, B]$ where $R >> A, B$. For example, we can take $A = 3, B = 2$ and $R = 10$ and formulate the finite element technique for this problem. The finite element formulation for this problem is detailed in the next section.

4 Finite element formulation

In this paper, the rectangular cross section of the waveguide is divided into a number of finite elements. An element is considered to be first order triangular in shape. Within each elemental triangle, the potential V is expanded as a linear combination of three affine linear functions with coefficients being the vertex potentials. The action functional $S[V]$ is computed by substituting this expansion to give a quadratic function of the vertex potentials having the form

$$\mathbf{V}^T A \mathbf{V} - k^2 \mathbf{V}^T B \mathbf{V} \quad (14)$$

Where $\underline{V}$ is a vector of vertex potentials and the matrices A, B are obtained from integrals of the form

$$\int \phi_i \phi_j dx.dy \qquad \text{and} \qquad \int \overrightarrow{\nabla} \phi_i, \overrightarrow{\nabla} \phi_j dx.dy$$

over the elemental triangles. The computational example in the preceding section gives numerical values for entries of the matrices A and B. Minimization of the quadratic form (Eq. 14) over $\mathbf{V}$ gives the generalized eigenvalue problem:

$$(\mathbf{A} - k^2 \mathbf{B})\mathbf{V} = 0$$

Therefore k^2 can be computed by solving the determinantal equation

$$|\mathbf{A} - k^2 \mathbf{B}| = 0$$

This expression has been used to obtain the modes (k^2) in the computations that follow. A schematics of a triangular finite element in the rectangular waveguide is shown in Fig. 1.

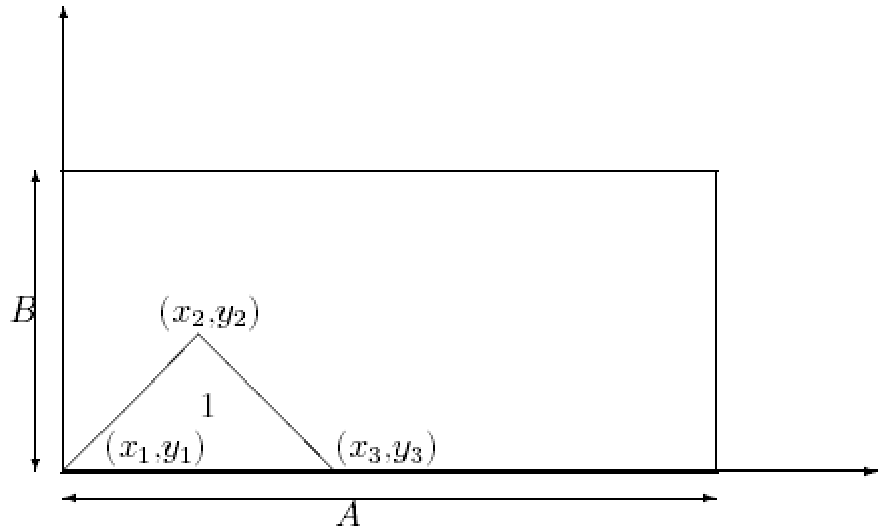

Figure 1: Schematic representation of a triangular finite element.

Consider a triangle having vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. Two vectors, $\overrightarrow{u}$ and $\overrightarrow{v}$ have been drawn by joining the vertices $[(x_1, y_1), (x_2, y_2)]$ and $[(x_1, y_1)$ and $(x_3, y_3)]$, respectively.

Let

$$d_1 = |u| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{15}$$

and

$$d_2 = |v| = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} \tag{16}$$

The unit vector along the two directions u and v are

$$\hat{u} = \frac{u}{|u|} = \frac{(x_2 - x_1, y_2 - y_1)}{d_1} \tag{17}$$

and

$$\hat{v} = \frac{v}{|v|} = \frac{(x_3 - x_1, y_3 - y_1)}{d_2} \tag{18}$$

any point (x, y) inside this triangle can be represented as

$$\begin{aligned}(x, y) &= (x_1, y_1) + u.\hat{u} + v.\hat{v} \\ &= (x_1, y_1) + \frac{u(x_2 - x_1, y_2 - y_1)}{d_1} + \frac{v(x_3 - x_1, y_3 - y_1)}{d_2}\end{aligned}$$

so

$$x = x_1 + \frac{u(x_2 - x_1)}{d_1} + \frac{v(x_3 - x_1)}{d_2} \tag{19}$$

and

$$y = y_1 + \frac{u(y_2 - y_1)}{d_1} + \frac{v(y_3 - y_1)}{d_2} \tag{20}$$

The solution of these two linear equations results in the variables u, v as linear functions of x, y. The area measure is given by

$$ds(u, v) = |\vec{u} \times \vec{v}| du.dv$$

where

$$|\vec{u} \times \vec{v}| = \sin \alpha$$

Here, α, the angle between the vectors u and v, is defined as

$$\begin{aligned}\cos \alpha &= \frac{u.v}{d_1.d_2} \\ &= \frac{(x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1)}{d_1.d_2}\end{aligned} \tag{21}$$

The integral of a function ϕcan be evaluated as

$$\begin{aligned}I(\phi) &= \frac{1}{2}\int_0^{d_1}\int_0^{d_2} \phi\left[x_1 + \frac{u(x_2 - x_1)}{d_1} + \frac{v(x_3 - x_1)}{d_2},\right. \\ &\left. y_1 + \frac{u(y_2 - y_1)}{d_1} + \frac{v(y_3 - y_1)}{d_2}\right] \sin \alpha.dudv\end{aligned} \tag{22}$$

For $\phi = 1$, we get

$$I(1) = \frac{d_1 d_2 \sin \alpha}{2} \tag{23}$$

which represents the area of the triangle.
Suppose we write

$$V(x, y) = ax + by + c \qquad \text{for} \qquad x, y \in \Delta$$

with Δ as the area bounded by the triangle. The constants a, b, c are chosen such that V at the vertices are given as follows:

$$V(x_1, y_1) = V_1,$$

$$V(x_2, y_2) = V_2,$$

$$V(x_3, y_3) = V_3.$$

Thus,

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}.$$

Now we find that

$$a = \frac{V_1(y_2 - y_3) + V_2(y_3 - y_1) + V_3(y_1 - y_2)}{\Delta}, \tag{24}$$

$$b = \frac{V_1(x_2 - x_3) + V_2(x_3 - x_1) + V_3(x_1 - x_2)}{\Delta}, \tag{25}$$

$$c = \frac{V_1(x_2y_3 - x_3y_2) + V_2(x_3y_1 - x_1y_3) + V_3(x_1y_2 - x_2y_1)}{\Delta} \tag{26}$$

where

$$\Delta = x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1. \tag{27}$$

So for $x, y \in \Delta$ we have

$$\begin{aligned} V(x, y) &= ax + by + c \\ &= V_1\phi_1(x, y) + V_2\phi_2(x, y) + V_3\phi_3(x, y) \end{aligned}$$

where

$$\phi_1(x, y) = \frac{(y_2 - y_3)x + (x_2 - x_3)y + (x_2y_3 - x_3y_2)}{\Delta}, \tag{28}$$

$$\phi_2(x, y) = \frac{(y_3 - y_1)x + (x_3 - x_1)y + (x_3y_1 - x_1y_3)}{\Delta}, \tag{29}$$

$$\phi_3(x, y) = \frac{(y_1 - y_2)x + (x_1 - x_2)y + (x_1y_2 - x_2y_1)}{\Delta}. \tag{30}$$

The following two integrals occur when one uses the finite element method

$$\textbf{First} \qquad I_1 = \int_\Delta V^2(x,y)dxdy$$

$$\textbf{Second} \qquad I_2 = \int_\Delta |\nabla V|^2 dxdy$$

These integrals are evaluated by dividing the rectangular waveguide cross-section into 28 triangular elements. The finite element mesh is shown in Fig. 2.

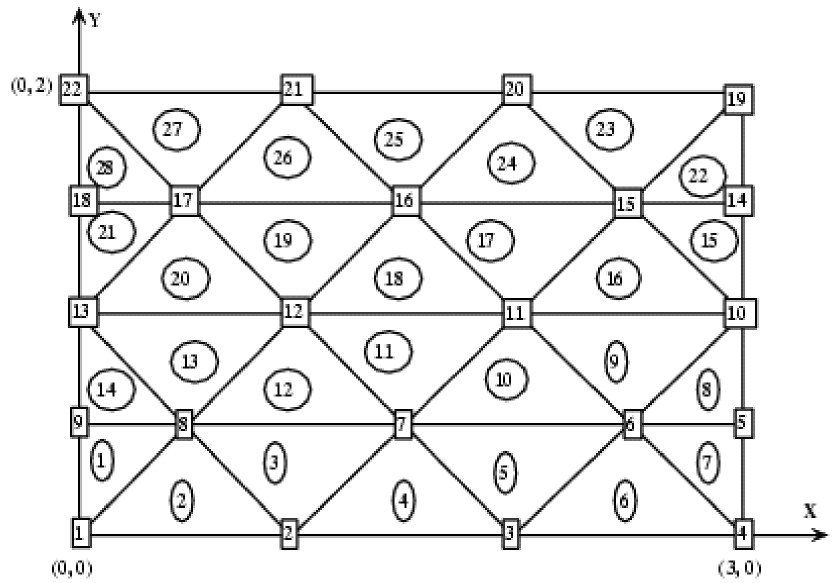

Figure 2: A finite element mesh (28 elements and 22 nodes). The numbers shown in circles and squares represent the nodes and elements, respectively.

The nodal coordinates of each elements are given in the Table 1.

Table 1: Nodal coordinates of the finite element mesh of Fig. 2

S.No.	Element no.	Coordinates
1	Element 1	(0.0,0.0),(0.5,0.5),(0.0,0.5)
2	Element 2	(0.0,0.0),(1.0.0.0),(0.5,0.5)
3	Element 3	(1.0,0.0),(1.5,0.5),(0.5,0.5)
4	Element 4	(1.0,0.0),(2.0,0.0),(1.5,0.5)
5	Element 5	(2.0,0.0),(2.5,0.5),(1.5,0.5)
6	Element 6	(2.0,0.0),(3.0,0.0),(2.5,0.5)
7	Element 7	(3.0,0.0),(3.5,0.5),(2.5,0.5)
8	Element 8	(2.5,0.5),(3.0,0.5),(3.0,1.0)
9	Element 9	(2.5,0.5),(3.0,1.0),(2.0,1.0)
10	Element 10	(1.5,0.5),(2.5,0.5),(2.0,1.0)
11	Element 11	(1.5,0.5),(2.0,1.0),(1.0,1.0)
12	Element 12	(0.5,0.5),(1.5,0.5),(1.0,1.0)
13	Element 13	(0.5,0.5),(1.0,1.0),(0.0,1.0)
14	Element 14	(0.0,0.5),(0.5,0.5),(0.0,1.0)
15	Element 15	(0.0,1.0),(3.0,1.5),(2.5,1.5)
16	Element 16	(2.0,1.0),(3.0,1.0),(2.5,1.5)
17	Element 17	(2.0,1.0),(2.5,1.5),(1.5,1.5)
18	Element 18	(1.0,1.0),(2.0,1.0),(1.5,1.5)
19	Element 19	(1.0,1.0),(1.5,1.5),(0.5,1.5)
20	Element 20	(0.0,1.0),(1.0,1.0),(0.5,1.5)
21	Element 21	(0.0,1.0),(0.5,1.5),(0.0,1.5)
22	Element 22	(2.5,1.5),(3.0,1.5),(3.0,2.0)
23	Element 23	(2.5,1.5),(3.0,2.0),(2.0,2.0)
24	Element 24	(1.5,1.5),(2.5,1.5),(2.0,2.0)
25	Element 25	(1.5,1.5),(2.0,2.0),(1.0,2.0)
26	Element 26	(0.5,1.5),(1.5,1.5),(1.0,2.0)
27	Element 27	(0.5,1.5),(1.0,2.0),(0.0,2.0)
28	Element 28	(0.0,1.5),(0.5,1.5),(0.0,2.0)

4.1 Acceleration term: integral $V^2(x,y)$

In the special case when the gravitational field is absent, V satisfies the Helmholtz equation

$$(\nabla^2 + k^2)V = 0 \tag{31}$$

which can be derived from the variational principle,

$$\delta\left[\int |\nabla V|^2 dxdy - k^2 \int V^2 dxdy\right] = 0 \tag{32}$$

The integral of V^2 term results in the k^2V term of the Helmholtz equation, which corresponds to the acceleration term $\frac{\partial^2 V}{\partial t^2}$ in the standard wave equation, when the wave field varies sinusoidally with time. Thus it is apt to call $\int V^2 dxdy$, the acceleration term in the action. The expression $\int |\nabla V|^2 dxdy$ in the action represents the field energy. Indeed in the static case, the action reduces to $\int |\nabla V|^2 dxdy$, which is proportional to the electrostatic field energy. Thus it is apt to view $\int |\nabla V|^2 dxdy$ as the field energy. It may be more apt to look upon $\int |\nabla V|^2 dxdy$ as the potential energy of the wave field and $\int V^2 dxdy$ as the kinetic energy for obvious reasons. Now

$$I_1 = \int_\Delta V^2(x,y)dxdy = \int_\Delta (ax+by+c)^2 \left(1+\frac{2U}{C^2}\right)^{-1/2} dxdy \tag{33}$$

By substituting the value of (x,y) in terms of (x_i, y_i) in equation (31), we get

$$\begin{aligned}\int_\Delta V^2(x,y)dxdy &= \frac{\sin\alpha}{2}\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} \left(1+\frac{2U}{C^2}\right)^{-1/2} \\ &\times \left[a\left(x_1+\frac{u(x_2-x_1)}{d_1}+\frac{v(x_3-x_1)}{d_2}\right)\right. \\ &+ \left. b\left(y_1+\frac{u(y_2-y_1)}{d_1}+\frac{v(y_3-y_1)}{d_2}\right)+c\right]^2 dudv\end{aligned} \tag{34}$$

Now separating the variable of u and v

$$\begin{aligned}\int_\Delta V^2(x,y)dxdy &= \frac{\sin\alpha}{2}\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} \left(1+\frac{2U}{C^2}\right)^{-1/2} \\ &\times \left[u\left(\frac{a(x_2-x_1)+b(y_2-y_1)}{d_1}\right)\right. \\ &+ \left. v\left(\frac{a(x_3-x_1)+b(y_3-y_1)}{d_2}\right)+c'\right]^2\end{aligned} \tag{35}$$

where

$$c' = ax_1 + by_1 + c.$$

We can write equation (33) as follows:

$$\int_\Delta V^2(x,y)dxdy = T_1+T_2+T_3+T_4+T_5+T_6 \tag{36}$$

where, the integrals (T_1 to T_6) are given as follows. These integrals have to be calculated for each pixel.

$$T_1 = \frac{\sin\alpha}{2}\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} [\frac{a(x_2-x_1)+b(y_2-y_1)}{d_1}]^2 \left(1+\frac{2U}{C^2}\right)^{-1/2} u^2 dudv \tag{37a}$$

$$T_2 = \frac{\sin\alpha}{2}\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} 2[\frac{a(x_2-x_1)+b(y_2-y_1)}{d_1}] \times[\frac{a(x_3-x_1)+b(y_3-y_1)}{d_2}]\left(1+\frac{2U}{C^2}\right)^{-1/2} uvdudv \tag{37b}$$

$$T_3 = \frac{\sin\alpha}{2}\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} [\frac{a(x_3-x_1)+b(y_3-y_1)}{d_2}]^2\left(1+\frac{2U}{C^2}\right)^{-1/2} v^2dudv \tag{37c}$$

$$T_4 = \frac{\sin\alpha}{2}\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} [\frac{2C'(a(x_2-x_1)+b(y_2-y_1))}{d_1}]\left(1+\frac{2U}{C^2}\right)^{-1/2} ududv \tag{37d}$$

$$T_5 = \frac{\sin\alpha}{2}\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} [\frac{2C'(a(x_3-x_1)+b(y_3-y_1))}{d_2}]\left(1+\frac{2U}{C^2}\right)^{-1/2} vdudv \tag{37e}$$

$$T_6 = \frac{\sin\alpha}{2}\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} C'^2\left(1+\frac{2U}{C^2}\right)^{-1/2} dudv \tag{37f}$$

Here

$$U(x,y) = -\frac{GM}{((x-R)^2+y^2)^{1/2}}$$

Let $GM = 1$ and velocity of light, C is given by $C = 1$ for simplification of calculation. Now

$$\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} V(x,y)^2dxdy = I(\phi) = T_1+T_2+T_3+T_4+T_5+T_6$$

The above integration for first element is given as follows:

$$\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} V(x,y)^2dxdy = 0.7072v_1^2+0.02v_2^2+0.4764v_3^2-0.722v_1v_2 - 1.5754v_1v_3-0.2026v_2v_3 \tag{38}$$

After calculating the above integrals for each element in the above stated manner, we will find the sum of these integrals over the elements in which we have divided the cross-section. Here we have divided the cross-section into 28 elements (Fig. 2). Summation of these integrals will result in a matrix B of size 22×22.

4.2 Acceleration term: integral $|\nabla V|^2$

$$\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} |\nabla V|^2 dudv = \int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} \left[1+\frac{2U}{C^2}\right] \times \left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right] dxdy \qquad (39)$$

Here,

$$dxdy = Jdudv \qquad (40)$$

The Jacobian J is given by

$$J = \begin{pmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{pmatrix} = \begin{pmatrix} \frac{x_2-x_1}{d_1} & \frac{x_3-x_1}{d_2} \\ \frac{y_2-y_1}{d_1} & \frac{y_3-y_1}{d_2} \end{pmatrix}.$$

Now

$$dx.dy = \frac{(x_2-x_1)(x_3-x_1)}{d_1.d_2} du.dv = -0.1250.$$

Finally

$$\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} |\nabla V|^2 dudv = \int_0^{0.25}\int_0^{0.125-0.5u} \left[1+\frac{2U}{C^2}\right] \times \left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right] \frac{(x_2-x_1)(x_3-x_1)}{d_1 d_2} dudv \qquad (41)$$

Here

$$\left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right] = a^2 + b^2 = 4(v_1^2 + v_2^2 + 2v_3^2 - 2v_2v_3 - 2v_1v_3)$$

After substituting all the values in equation (44) and integrate, we get

$$\int_0^{d_1}\int_0^{d_2-\frac{d_2 u}{d_1}} |\nabla V|^2 dudv = -0.0072(v_1^2 + v_2^2 + 2v_3^2 - 2v_2v_3 - 2v_1v_3)$$

Here v_1,v_2, $v_3 \cdots v_n$ are the nodal potential. Solution of integration of $|\nabla V|^2 dudv$ for all the 28 element, computed in same manner will result in a matrix A of size 22×22.

5 Results

Simulations were carried out using MATLAB and Maple softwares for a rectangular ($0.1 \leq b/a \leq 1$) waveguide. The numerical values were obtained with and without gravitational effects. The frequency of propagation (k^2) for the waveguide in the gravitational field is obtained, as shown in the Fig. 3. We can see from the graph [Fig. 3] that the modes get shifted due to the effect of gravitation.

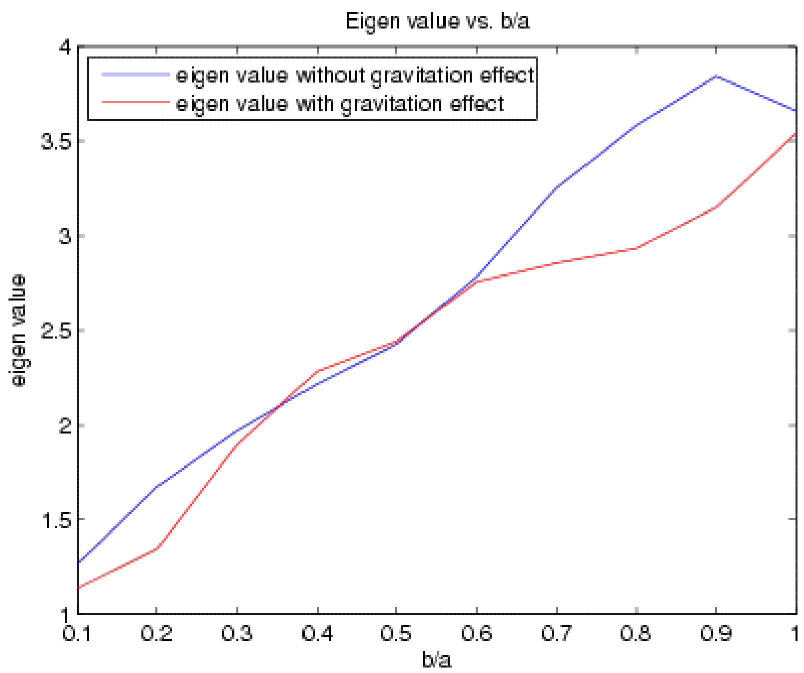

Figure 3: Computed eigenvalues.

Also shown in the figure are the frequencies of propagation for the waveguide without the gravitational effect. A comparison of the two values (with and without gravitational effects) clearly shows a shift in the propagation modes. The gravitational field exerts its influence on the propagating electromagnetic wave via a finite space-time curvature. This is modelled by a metric with spatially varying coefficients. When the covariant scalar wave equation is written down in such a metric then coupling terms arise involving both the metric coefficients and the propagating field. These terms result in a small shift in the modes, which has been analyzed in this study. These results may be useful in the design process of rectangular, square and ridge waveguide.

6 Concluding remarks

A finite element formulation for the computation of the propagation modes of the electromagnetic waveguide with and without gravitational effects has been presented and discussed. In presence of the field of gravitation, a shift in the modes of the waveguide is noticed [11]. Textbook analysis of waveguide does not take into account the effect of gravitation on the propagation of electromagnetic waves. The analysis presented in this paper does take into account this interaction which can be employed to analyze the nature of a massive distribution of matter when one space-craft passes close to it by looking at the behavior of a waveguide inside the space-craft.

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