HAIT Journal of Science and Engineering C, Volume 4, Issues 1-2, pp. 109-127

Extrapolation of multi-dimensional Fourier signals

Lev Aizenberg, Igor Grossman*, and Irina Volinsky

Department of Mathematics, Bar-Ilan University,
Ramat-Gan 52900, Israel
* *Corresponding author: matrim1976@yahoo.com*
Received 4 July 2004, accepted 5 March 2006

Abstract

There is a well-known method of extrapolation of signals with the finite Fourier spectrum developed by Gershberg and Papoulis. Aizenberg introduced a method which uses the theory of Hardy spaces in complex analysis. In a recent publication, a new extrapolation method for one-dimensional signals was proposed. The method is based on combining the methods of Aizenberg and Gershberg-Papoulis and sometimes permits optimized calculations (after a certain regularization, if needed). This method was extended to both two-dimensional and three-dimensional signals.

1 Basic definitions

The Wiener class of functions. The Wiener class W^3_α is the class of functions in $L^2(R^3)$ that have the Fourier transform (spectrum)

$$g(w) = \int_{R^3} f(x)e^{-i(w_1x_1+w_2x_2+w_3x_3)}dx \ ,$$

whose support is concentrated in the parallelepiped $\{w : |w_j| \leq \alpha_j, \text{ j} = 1,2,3\}$.

We also introduce the class $W^3_{\alpha+}$ of functions in $L^2(R^3)$ whose support of the spectrum is in $\{w : 0 \leq w_j \leq a_j, \text{ j} = 1,2,3\} \subset R^3_+ = \{w : w_j \geq 0,$

j = 1,2,3}. Sometimes we will denote the class W^3_α by $L^2_{-\alpha,\alpha}(R^3)$ and $W^3_{\alpha+}$ by $L^2_{0,\alpha}$ (R^3).

The Hardy class of functions. The Hardy class of functions $H^2(D_\delta)$ (or H^2_δ) is the class of functions holomorphic in the product of half-planes D_δ = {z : Im z_j > -δ, j = 1, 2, 3} $\subset$ C ^{3}. Here and below, δ is a fixed positive constant. $H^2(D_\delta)$ is the subspace of $A(D_\delta)$ (space of functions, holomorphic in D_δ) that consist of functions satisfying the condition $\int\limits_{R^3} |f(x+iy)|^2 dx \leq C$ where -δ < y_j < ∞, j = 1, 2, 3.

The Cauchy transform. The Cauchy transform (of $\varphi \in L^2(R)$) is defined as

$$C_{2\delta}(\varphi)(z) = \int\limits_{-\tau}^{+\tau} \frac{\varphi(t)}{z - t + 2i\delta} dt$$

2 Formulation of the problem

Consider a famous problem in the theory of Fourier signals: a signal with a finite Fourier spectrum is an entire function. If one can extrapolate Fourier signals properly, then it is possible to achieve super resolution of physical devices, control of narrow band noise, and so on.

Those 3-dimensional signals are the functions of the Wiener class W^3_α. Note the connection between the Wiener classes W^3_α and $W^3_{\alpha+}$: If f $\in W^3_\alpha$, then $f(z)e^{i<\alpha,z>} \in W^3_{2\alpha+}$ and, conversely, if f $\in W^3_{2\alpha+}$, then $f(z)e^{-i<\alpha,z>} \in W^3_\alpha$. Further, $W^3_{\alpha+} \subset H^2(D_\delta)$ for all $\delta > 0$.

3 The Aizenberg method

We can transfer our problem of extrapolation of Fourier signals into the framework of Hardy spaces and concentrate on the problem of extrapolation of Hardy class functions. It is shown in [4] that: For every function f (z) $\in H^2(D_\delta)$ the following equations are true (the convergence is that uniform on compact sets in D_δ and, moreover, in norm of $H^2(D_\delta)$):

for n = 1:

$$f(z) = \lim_{m\to\infty} \sum_{k=1}^{m} f(x_k) \tfrac{2i\delta}{z-x_k+2i\delta} \prod_{j=1\ j\neq k}^{m} \tfrac{(z-x_j)(x_k-x_j+2i\delta)}{(z-x_j+2i\delta)(x_k-x_j)}$$

for n = 2:

$$f(z_1,z_2)=\lim_{m\to\infty}\sum_{k_1,k_2=1}^{m} f(x_{1k_1},x_{2k_2})\frac{(2i\delta)^2}{(z_1-x_{1k_1}+2i\delta)(z_2-x_{2k_2}+2i\delta)}\times$$

$$\times\prod_{j=1\ j\neq k_1}^{m}\frac{(z_1-x_{1j})(x_{1k_1}-x_{1j}+2i\delta)}{(z_1-x_{1j}+2i\delta)(x_{1k_1}-x_{1j})}\prod_{j=1\ j\neq k_2}^{m}\frac{(z_2-x_{2j})(x_{2k_2}-x_{2j}+2i\delta)}{(z_2-x_{2j}+2i\delta)(x_{2k_2}-x_{2j})}$$

for n = 3:

$$f(z_1,z_2,z_3)=\lim_{m\to\infty}\sum_{k_1,k_2,k_3=1}^{m} f(x_{1k_1},x_{2k_2},x_{3k_3})\times$$

$$\times\frac{(2i\delta)^2}{(z_1-x_{1k_1}+2i\delta)(z_2-x_{2k_2}+2i\delta)(z_3-x_{3k_3}+2i\delta)}\times\prod_{j=1\ j\neq k_1}^{m}\frac{(z_1-x_{1j})(x_{1k_1}-x_{1j}+2i\delta)}{(z_1-x_{1j}+2i\delta)(x_{1k_1}-x_{1j})}\times$$

$$\times\prod_{j=1\ j\neq k_2}^{m}\frac{(z_2-x_{2j})(x_{2k_2}-x_{2j}+2i\delta)}{(z_2-x_{2j}+2i\delta)(x_{2k_2}-x_{2j})}\prod_{j=1\ j\neq k_3}^{m}\frac{(z_3-x_{3j})(x_{3k_3}-x_{3j}+2i\delta)}{(z_3-x_{3j}+2i\delta)(x_{3k_3}-x_{3j})}$$

where x_n and x_{1n}, x_{2n}, x_{3n} are the sets of points where the value of function f is known. Variants of these formulas for higher dimension are considered in [4] as well.

4 The Gershberg-Papoulis method

4.1 The original Gershberg-Papoulis method

Let us consider the space $H=L^2(R)$. Let I_1 be the canonical injection of the subspace $H_1= L^2_{0,2\sigma}(R)$ in $L^2(R)$; and I_2 the canonical injection of $H_2 = L^2([-\tau,\tau])$ in $L^2(R)$, here for every $\varphi \in H_2$ we choose its corresponding extension to zero. Then P_1: $H \to H_1$, P_2: $H \to H_2$ are orthogonal projections. The operator $\beta = P_2 \circ I_1$ from P_2 to subspace H_1 has the adjoint β^* $= P_1 \circ I_2$, and the operator $\beta^*\beta$ has the analytic representation:

$$\beta*\beta(f)(x)=\int_{-\tau}^{+\tau} e^{i\sigma(x-t)}\frac{\sin\sigma(x-t)}{\pi(x-t)}f(t)dt,\ \text{where } x\in R,\ f\in H_1.$$

The following algorithm is described to make it possible to introduce the parameter δ to reduce the problem to that within the framework of Hardy spaces. Let us consider the operator B_0: $H_1 \to H_1$ defined by $B_0\ (\varphi) = \varphi + (\beta^*\beta)(f - \varphi) = \varphi + P_1P_2\ (f - \varphi)$, $\varphi \in H_1$, where f is given, and only the part of it on the interval $[-\tau, \tau]$ is known. It is shown in [1] that the only possible fixed point of this operator is the extrapolation of the given function. Now, the iteration is introduced:

$$g_{n+1} = B_0(g_n), \text{ with } g_0 = P_1P_2(f) = \beta^*\beta(f);$$

and the analytic expression for g_n is $g_n = (Id_{H1} - (Id_{H1} - P_1P_2)^{n+1})(f)$, where Id is the identity operator.

4.2 The Gershberg-Papoulis method in Hardy spaces

In this method, a new operator is introduced: B_δ: $L^2_{0,2\sigma}(R) \rightarrow L^2_{0,2\sigma}(R)$, which uses the Cauchy transformation $C_{2\delta}$. Let i^δ_σ be the canonical injection from $L^2_{0,2\sigma}(R)$ in H^2_δ, and let its adjoint operator be Π^δ_σ. Now, let us define an operator α^δ_σ from $L^2_{0,2\sigma}(R)$ to $L^2([-\tau,\tau])$ as $\alpha^\delta_\sigma = R_\tau \circ i^\delta_\sigma$, where $R_\tau : H^2_\delta \rightarrow L^2([-\tau, \tau])$ is an operator of restriction $\varphi \rightarrow \varphi_{|[-\tau,\tau]}$. Also, the adjoint operator may be defined as $(\alpha^\delta_\sigma)^* = \Pi^\delta_\sigma \circ R^*_\tau = -\frac{1}{2i\pi}\Pi^\delta_\sigma \circ C_{2\delta}$.

It is now possible to define the operator $B_\delta : L^2_{0,2\sigma}(R) \rightarrow L^2_{0,2\sigma}(R)$ as $B_\delta(\varphi) = \varphi + (\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma(f - \varphi)$, $\varphi \in L^2_{0,2\sigma}(R)$.

Note that, exactly as with the operator B_0, the definition requires only the knowledge of the given function f on a certain interval $[-\tau, \tau]$. Also, $(\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma(\varphi)(z) = P_1P_2(\varphi)(z + 2i\delta)$, $\varphi \in L^2_{0,2\sigma}(R)$.We now define the iteration: $g_{n+1} = B_\delta (g_n)$, with $g_0 = ((\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma)(f)$; the analytic expression for g_n is

$$g_n = (Id_{L^2_{0,2\delta}(R)} - (Id_{L^2_{0,2\delta}(R)} - (\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma)^{n+1})(f).$$

It is shown in [1] that the only possible fixed point of this iteration is the extrapolation of the given function. The research shows that both the Gershberg-Papoulis and the new method of French mathematicians work in the three-dimensional space.

4.3 The Gershberg-Papoulis method in the two- and three-dimensional spaces

The method of Gershberg-Papoulis can be extended into two- and three-dimensional cases. Consider all the operators to be two-dimensional, and the analytic expression of the operator $\beta^*\beta$ will take the form

$$\beta * \beta(f)(x,y) = \iint\limits_{\tau_1,\tau_2} e^{i\sigma_1(x-t)+i\sigma_2(y-s)} \frac{\sin\sigma_1(x-t)}{\pi(x-t)} \cdot \frac{\sin\sigma_2(y-s)}{\pi(y-s)} f(s,t)dsdt,$$

where x, y $\in$ R, f $\in H^2_1$ – the H_1 space of functions of two variables.

Since our function f $\in H_1^2$, we can change the double integral form to repeated, thus modifying the expression to the following:

$$\beta*\beta(f)(x,y) = \int_{-\tau_1}^{+\tau_1} e^{i\sigma_1(x-t)} \frac{\sin\sigma_1(x-t)}{\pi(x-t)} \left(\int_{-\tau_2}^{+\tau_2} e^{i\sigma_2(y-s)} \frac{\sin\sigma_2(y-s)}{\pi(y-s)} f(s,t)ds \right) dt$$

where x, y $\in$ R, f $\in H_1^2$.

Now, to redefine the iteration let us consider the operator B_0^2: $H_1^2 \to H_1^2$ defined by B_0^2 (φ) = φ + ($\beta^*\beta$)(f - φ) = φ + P_1P_2 (f - φ), $\varphi \in H_1^2$, where f is given, with only the part of it on [-τ_1, τ_1]$\times$[-τ_2, τ_2] known. We can prove that the only possible fixed point of this operator is f.

Proof: Consider $\varphi \in H_1^2$ such that $\varphi = B_0^2$ (φ). Then $\varphi = \varphi + P_1P_2$ (f - φ), and, consequently, P_1P_2 (f - φ) = 0, which means that φ = f (since the operator P_1P_2 is injective). We now introduce the new iteration: g_{n+1} = ${B_0}^2$ (g_n), with g_0 = P_1P_2 (f) = $\beta^*\beta$ (f) (two-dimensional case); and the analytic expression for g_n is $g_n = (Id_{H_1^2} - (Id_{H_1^2} - P_1P_2)^{n+1})(f)$
The only possible fixed point of this iteration is f.

A similar treatment is true for the three-dimensional spaces. In that case, the analytic expression of the operator $\beta^*\beta$ is of the form

$$\beta^*\beta(f) = \int_{-\tau_1}^{+\tau_1} e^{i\sigma_1(x-t)} \frac{\sin\sigma_1(x-t)}{\pi(x-t)} \times$$
$$\times \left(\int_{-\tau_2}^{+\tau_2} e^{i\sigma_2(y-s)} \frac{\sin\sigma_2(y-s)}{\pi(y-s)} \left(\int_{-\tau_3}^{+\tau_3} e^{i\sigma_3(w-h)} \frac{\sin\sigma_3(w-h)}{\pi(w-h)} f(h,s,t)dh \right) ds \right) dt,$$

where x, y, w $\in$ R, f $\in H_1^3$.

5 A new method

5.1 A new method in the two- and three-dimensional spaces

A new operator is introduced: B_δ^2: $L_{0,2\sigma}^2(R^2) \to L_{0,2\sigma}^2(R^2)$, which uses the Cauchy transformation $C_{2\delta}^2$ defined as

$$C_{2\delta}^2(\varphi)(z_1,z_2) = \iint_{\tau_1,\tau_2} \frac{\varphi(t,s)}{(z_1-t+2i\delta)(z_2-s+2i\delta)} dtds.$$

Let i_σ^δ be a canonical injection from $L_{0,2\sigma}^2(R^2)$ in H_δ^2, with the adjoint operator Π_σ^δ. Now, let us define an operator α_σ^δ from $L_{0,2\sigma}^2(R^2)$ to

$L^2([-\tau_1, \tau_1] \times [-\tau_2, \tau_2])$ by $\alpha_\sigma^\delta = R_\tau \circ i_\sigma^\delta$, where $R_\tau : H_\delta^2 \to L^2([-\tau, \tau])$ is the operator of restriction $\varphi \to \varphi_{|[-\tau_1,\tau_1]\times[-\tau_2,\tau_2]}$. Also, the adjoint operator may be defined as

$$(\alpha_\sigma^\delta)^* = \Pi_\sigma^\delta \circ R_\tau^* = -\tfrac{1}{2i\pi}\Pi_\sigma^\delta \circ C_{2\delta}.$$

It is now possible to define the operator B_δ^2: $L_{0,2\sigma}^2(R^2) \to L_{0,2\sigma}^2(R^2)$ by $B_\delta^2\ (\varphi) = \varphi + (\alpha_\sigma^\delta)^* \circ \alpha_\sigma^\delta\ (f - \varphi)$, $\varphi \in L_{0,2\sigma}^2(R^2)$.

Note that, just as it was with the operator B_0, the definition demands only knowledge of a given function f on a certain square $[-\tau_1, \tau_1]\times[-\tau_2, \tau_2]$. What we need to prove is that
$(\alpha_\sigma^\delta)^* \circ \alpha_\sigma^\delta\ (\varphi)(z_1, z_2) = P_1P_2\ (\varphi)(z_1 + 2i\delta, z_2+2i\delta)$, $\varphi \in L^2{}_{0,2\sigma}(R^2)$.

Proposition 1: For each function $\varphi \in H_\delta^2(R^2)$ the following equality takes place:

$$\left(\Pi_\sigma^\delta \circ C_{2\delta}^2 \circ R_{\tau_1,\tau_2}\right)(\varphi)(z_1, z_2) = -4 \iint\limits_{\tau_1,\tau_2} e^{i\sigma((z_1+2i\delta-u_1)+(z_2+2i\delta-u_2))}\times$$
$$\times\frac{\sin\sigma(z_1+2i\delta-u_1)\sin\sigma(z_2+2i\delta-u_2)}{(z_1+2i\delta-u_1)(z_2+2i\delta-u_2)}\varphi(u_1,u_2)du_1du_2,$$

where function φ is known only on $[-\tau_1, \tau_1]\times[-\tau_2, \tau_2]$.

Proof: Consider $F\ (z_1, z_2) \in H_\delta^2(R^2)$. This function is isometric to $F_1\ (\xi_1, \xi_2) = F\ (i(\xi_1\text{-}\delta), i(\xi_2\text{-}\delta)) \in H^2(\Pi^2)$, and the Laplace transform ϑ : f $\in L^2(R^2) \to \vartheta$ (f) $\in H^2(\Pi^2)$ is isometric as well. We can now build the bijection L: $L_2([0, +\infty]\times[0, +\infty]) \to H_\delta^2$, defined as $L(f)(z_1, z_2) = \vartheta\ (f)(\delta - iz_1, \delta - iz_2)$. The analytic expression for this transformation is

$$L(f)(z_1,z_2) = \int\limits_0^{+\infty}\int\limits_0^{+\infty} e^{-t_1\delta-t_2\delta}f(t_1,t_2)e^{it_1z_1+it_2z_2}dt_1dt_2.$$

The orthogonal projection p_σ of $L_2([0, +\infty]\times[0, +\infty])$ into $L^2([0, 2\sigma]\times[0, 2\sigma])$, which is $f \to \chi_{[0,2\sigma]\times[0,2\sigma]}f$, defines the orthogonal projection $\Pi_\sigma^\delta : H_\delta^2 \to L_{0,2\sigma}^2(R^2)$.

Consider $f \in L^2([0, 2\sigma][0, 2\sigma])$ and $x_1, x_2 \in R$, thus receiving:

$$L(f)(x_1,x_2) = \int\limits_0^{2\delta}\int\limits_0^{2\delta} e^{-t_1\delta-t_2\delta}f(t_1,t_2)e^{-it_1x_1-it_2x_2}dt_1dt_2 = \Im^*(e^{-\delta\bullet}f)(x_1,x_2),$$
$$L(f)(z_1,z_2) = \int\limits_0^{+\infty}\int\limits_0^{+\infty} e^{-t_1\delta-t_2\delta}f(t_1,t_2)e^{it_1z_1+it_2z_2}dt_1dt_2.$$

which is an element of $L^2_{0,2\sigma}(R^2)$. This element can be extended up to the entire function $L(f)(z_1,z_2)$, by replacing each x_1 with z_1 and each x_2 with z_2. Since the isometry preserves orthogonality, we conclude that $\Pi^\delta_\sigma \circ L = L \circ p_\sigma$, and that the image, on Π^δ_σ, of the function

$$g(z_1, z_2) = \int\limits_0^{+\infty}\int\limits_0^{+\infty} e^{-t_1\delta - t_2\delta} f(t_1, t_2) e^{it_1z_1 + it_2z_2} dt_1 dt_2$$

is the function

$$L(\chi_{[0,2\sigma]\times[0,2\sigma]} f)(z_1, z_2) = \int\limits_0^{2\sigma}\int\limits_0^{2\sigma} e^{-t_1\delta - t_2\delta} f(t_1, t_2) e^{itz_1 + itz_2} dt_1 dt_2.$$

Since $g = \mathfrak{I}^*(e^{-\delta\bullet} f)$,

$$\Pi^\delta_\sigma(g)(z_1, z_2) = \Im^*(\chi_{[0,2\sigma]\times[0,2\sigma]}(e^{-\delta\bullet} f))(z_1, z_2) = (e^{i\sigma\bullet}\tfrac{\sin\sigma\bullet}{\pi\bullet} g)(z_1, z_2).$$

Thus,

$$\Pi^\delta_\sigma(g)(z_1, z_2) =$$
$$\int\limits_R\int\limits_R e^{i\sigma((z_1-s_1)+(z_2-s_2))} \frac{\sin\sigma(z_1 - s_1)}{\pi(z_1 - s_1)} \frac{\sin\sigma(z_2 - s_2)}{\pi(z_2 - s_2)} g(s_1, s_2) ds_1 ds_2.$$

This is an analytic expression for Π^δ_σ, the extension of P_2, where δ is not included. Now, consider the function

$$g(z_1, z_2) = (C^2_{2\delta} \circ R_{\tau_1,\tau_2})(\varphi)(z1, z_2) =$$
$$\int\limits_{-\tau_1}^{\tau_1}\int\limits_{-\tau_2}^{\tau_2} \frac{\varphi(u_1, u_2)}{(z_1 - u_1 + 2i\delta)(z_2 - u_2 + 2i\delta)} du_1 du_2.$$

Using Fubini's theorem, we obtain:

$$\left(\Pi^\delta_\sigma C^2_{2\delta} R_{\tau_1,\tau_2}\right)(\varphi)(z_1, z_2) =$$
$$\int\limits_R\int\limits_R \left(\int\limits_{-\tau_1}^{\tau_1}\int\limits_{-\tau_2}^{\tau_2} \frac{\varphi(u_1, u_2)}{(s_1 - u_1 + 2i\delta)(s_2 - u_2 + 2i\delta)} du_1 du_2\right) \times$$
$$\times e^{i\sigma((z_1-s_1)+(z_2-s_2))} \frac{\sin\sigma(z_1 - s_1)}{\pi(z_1 - s_1)} \frac{\sin\sigma(z_2 - s_2)}{\pi(z_2 - s_2)} ds_1 ds_2 =$$
$$=$$
$$\int\limits_{-\tau_1}^{\tau_1}\int\limits_{-\tau_2}^{\tau_2} \varphi(u_1, u_2) \left\{\int\limits_R\int\limits_R \frac{e^{i\sigma((z_1-s_1)+(z_2-s_2))} \sin\sigma(z_1 - s_1) \sin\sigma(z_2 - s_2)}{\pi^2(z_1 - s_1)(z_2 - s_2)(s_1 - u_1 + 2i\delta)(s_2 - u_2 + 2i\delta)}\right\} du_1 du_2.$$

Elementary calculations, using the Cauchy formula, give that the value of the expression in brackets is

$$-4e^{i\sigma((z_1-u_1+2i\delta)+(z_2-u_2+2i\delta)}\frac{\sin\sigma(z_1-u_1+2i\delta)}{(z_1-u_1+2i\delta)}\frac{\sin\sigma(z_2-u_2+2i\delta)}{(z_2-u_2+2i\delta)},$$

which completes the proof of the proposition.

Corollary:

1) For each $\varphi \in H^2_\delta$, we have $\Pi_\sigma C^2_{2\delta} R_{\tau_1,\tau_2}(\varphi)(z_1, z_2)$=-$4\pi^2 P_1P_2(\varphi)(z_1+2i\delta$, $z_2+2i\delta)$, where z_1, $z_2 \in$ C. $P_1P_2(\varphi)$ is the orthogonal extension of R_{τ_1,τ_2} (φ)=$P_2(\varphi|R^2)$ on $L_{0,2\sigma}(R^2)$, a subspace of $L^2(R^2)$.

2) For each $\varphi \in L_{0,2\sigma}(R^2)$ $((\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma)(\varphi)(z_1, z_2)$=$P_1P_2(\varphi)(z_1+2i\delta$, $z_2+2i\delta)$, where z_1, $z_2 \in$ C.

Proof: This corollary is a consequence of the well-known fact that the orthogonal projection $P_1 : L^2(R^2) \to L^2_{0,2\sigma}(R^2)$ is

$$f \longmapsto \int\int e^{i\sigma((t_1-s_1)+(t_2-s_2))}\frac{\sin\sigma(t_1-s_1)}{\pi(t_1-s_1)}\frac{\sin\sigma(t_2-s_2)}{\pi(t_2-s_2)}f(s_1,s_2)ds_1ds_2.$$

Lemma 1:

1) The operator $(\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma$ = - $(1/4\pi^2)\Pi^\delta_\sigma \circ C^2_{2\delta} \circ R_{\tau_1,\tau_2} \circ i^\delta_\sigma$ is self-adjoint, compact, and injective.

2) For each f $\in L^2_{0,2\sigma}(R^2)$ (which is known only on $[-\tau_1,\tau_1]\times[-\tau_2,\tau_2]$), the operator B^2_δ: $\varphi \to \varphi + ((\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma)(f - \varphi)$ of the space $L^2_{0,2\sigma}(R^2)$ into itself has the only fixed point f.

Proof: The operator $(\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma$ is self-adjoint by definition, and compact, the latter follows from the analytic expression. Finally, if $P_1P_2(\varphi) = 0$, then $\varphi = 0$, since $P_1P_2(\varphi)$ is injective. Now, it is possible to define the iteration $g_{n+1} = B^2_\delta\ (g_n)$, with $g_0 = ((\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma)(f)$; and the analytic expression for g_n is $g_n = (Id_{L^2_{0,2\delta}(R)} - (Id_{L^2_{0,2\delta}(R)} - (\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma)^{n+1})(f)$.
The only possible fixed point of this iteration is f.

A similar discussion is true for three-dimensional spaces. In that case, operator $(\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma$ is of the form $(\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma(\varphi)(z_1,z_2,z_3) = P_1P_2(\varphi)(z_1+2i\delta$, $z_2+2i\delta$, $z_3+2i\delta)$, $\varphi \in L^2_{0,2\sigma}$ (R^3), where $z_1,z_2,z_3 \in$ R, f $\in H^3{}_1$.

5.2 Modification of the new method in the two- and three-dimensional spaces

Let us add an additional parameter λ and study the operator Id - $i\lambda(\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma$ for $(i\lambda)^{-1} \notin \{\lambda^\delta_k$, k $\geq 0\}$, and in particular, $\lambda \in R^*$. The operators $B^2_{0,\lambda}$ and $B^2_{\delta,\lambda}$ are defined as

$$B^2_{0,\lambda}\ (\varphi) = \varphi + i\lambda P_1 P_2\ (f - \varphi),\ \varphi \in L^2_{0,2\sigma}(R^2),\ \delta = 0,$$

and

$$B^2_{\delta,\lambda}\ (\varphi) = \varphi + i\lambda\ (\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma\ (f - \varphi),\ \varphi \in L^2_{0,2\sigma}(R^2),\ \delta > 0;$$

while new iterations are introduced for both operators, respectively, as follows:

$$g_{n+1} = (Id - i\lambda P_1 P_2)\ g_n + i\lambda P_1 P_2\ (f)$$

and

$$g_{n+1} = (Id - i\lambda(\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma)\ g_n + i\lambda(\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma(f).$$

It is possible to modify them into new iterations by using the inverse operators:

$$h_{n+1} = (Id - i\lambda P_1 P_2)^{-1}\ h_n + (Id - i\lambda P_1 P_2)^{-1}\ (i\lambda P_1 P_2\ (f))$$

and

$$h_{n+1} = (Id - i\lambda(\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma)^{-1}\ h_n - (Id - i\lambda(\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma)^{-1}\ (i\lambda(\alpha^\delta_\sigma)^* \circ \alpha^\delta_\sigma(f)).$$

It can be assumed that there are privileged pairs $(\delta,\ \lambda)$ that provide an opportunity to control the rate of convergence. It is shown in [1], that the best rate can be obtained (for the one-dimensional case) when $\delta = 0$. The same is true for the two- and three-dimensional cases.It is possible to introduce some regularization processes that use these parameters as well.

A similar discussion is true for three-dimensional spaces.

6 Computational part of the research

In our research, different types of computing experiments were performed: for both two and three-dimensional spaces. The experiments were performed on Matlab under UNIX and showed good results. Some of the resulting graphs can be found in Appendix A.

Appendix A

The following graphs show the extrapolation result for the function $f_2(x, y, z) = 7\frac{\sin(\frac{1}{2}x)\,\sin(3y)\,\sin(\frac{1}{2}z)}{\pi^3 xyz}$ from the cube [-1,1]×[-1,1]×[-1,1] to the cube [-8,8]×[-8,8]×[-8,8], by using the Gershberg-Papoulis method in three-dimensional space. The value of parameters: $\sigma_1 = 1/2$, $\sigma_2 = 3$, $\sigma_3 = 1/2$. We use plane sections to plot the original and extrapolated functions.

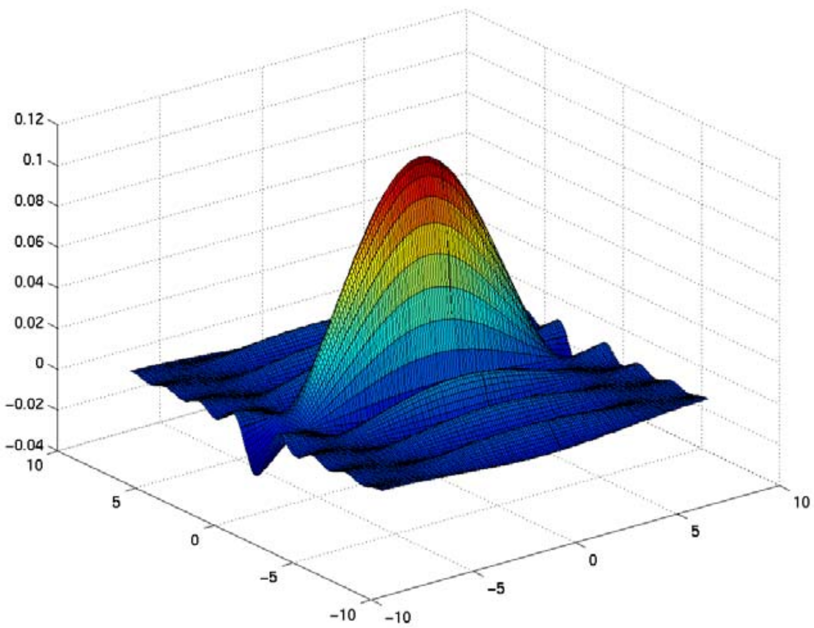

Figure 1: The original function in place section x = 3.

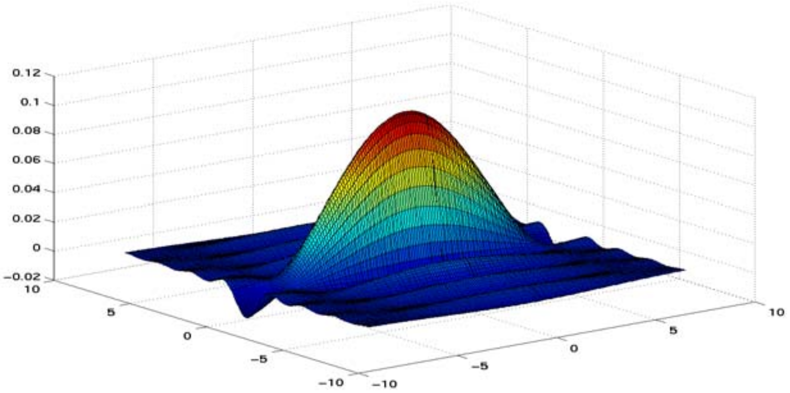

Figure 2: The extrapolated function in plane section x = 3.

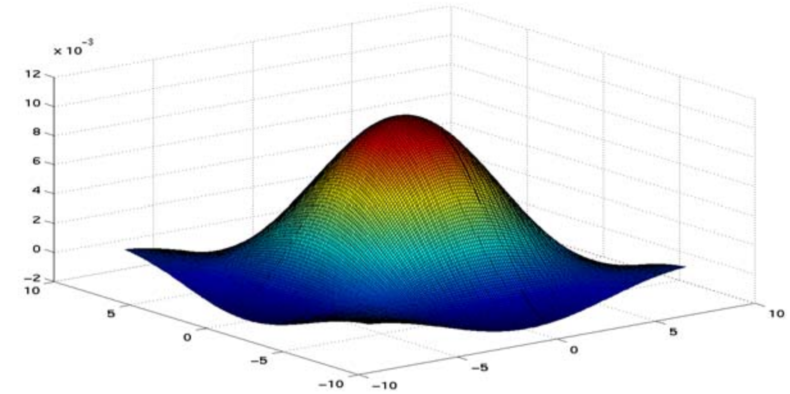

Figure 3: The original function in plane section y = 4.5.

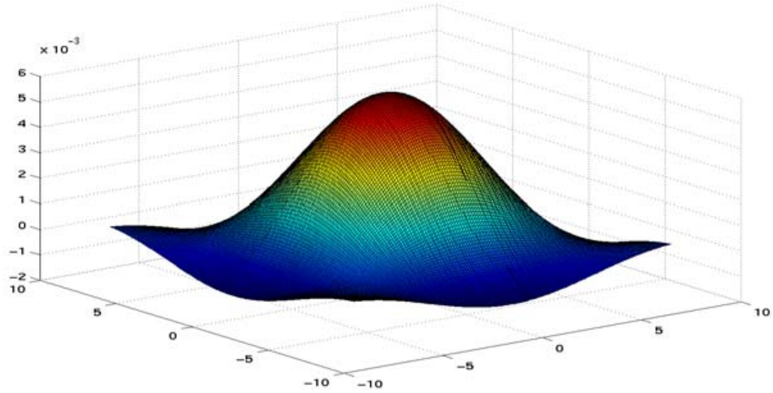

Figure 4: The extrapolated function in plane section y = 4.5.

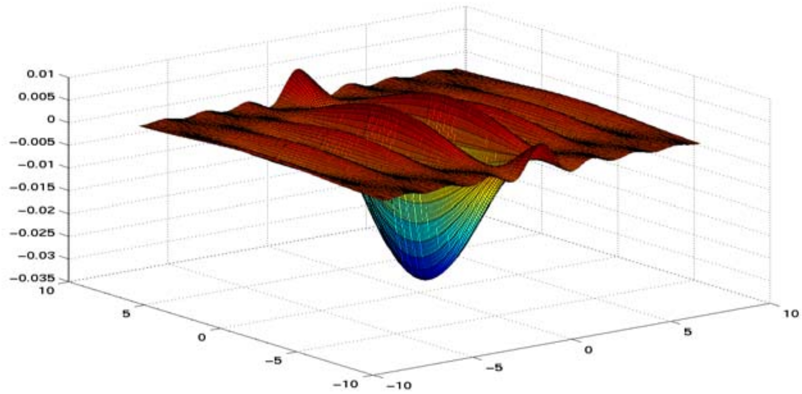

Figure 5: The original function in plane section z = 8.

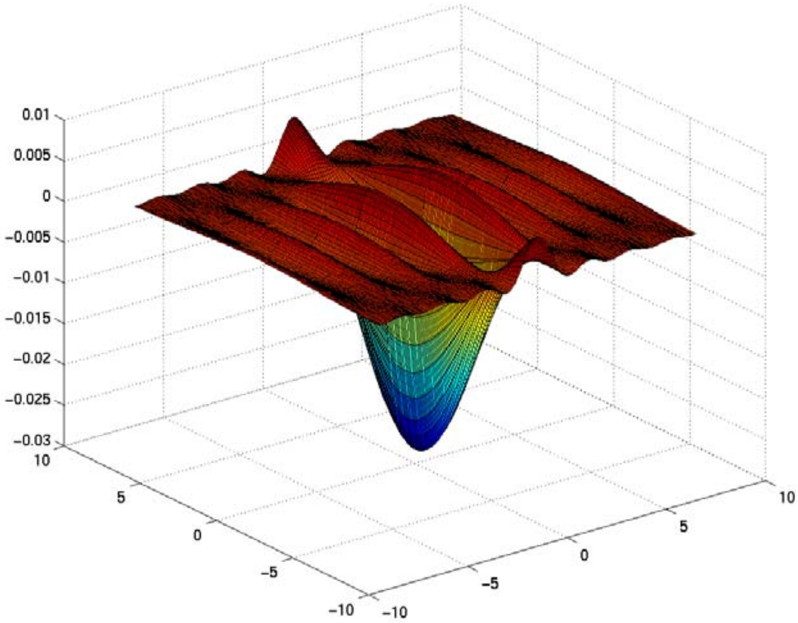

Figure 6: The extrapolated function in plane section z = 8.

The following graphs show the extrapolation result for the function $f_2(x,y,z) = 7\frac{\sin(\frac{1}{2}x)\,\sin(3y)\,\sin(\frac{1}{2}z)}{\pi^3\,x\,y\,z}$ from the cube [-1,1]×[-1,1]×[-1,1] to the cube [-8,8]×[-8,8]×[-8,8] using the new method in three-dimensional space. The value of parameters: $\sigma_1 = 1/2$, $\sigma_2 = 3$, $\sigma_3 = 1/2$. We created graphs for different δ values, the graphs show that the best results are obtained when δ is close to zero. We use plane sections to plot the original and extrapolated functions.

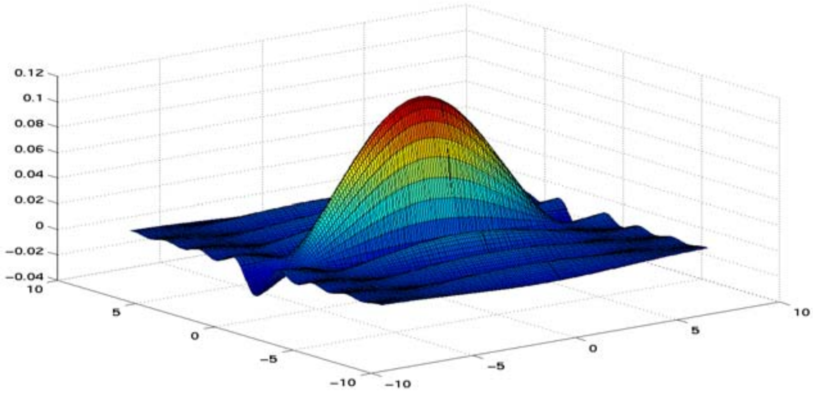

Figure 7: The original function in plane section x = 3.

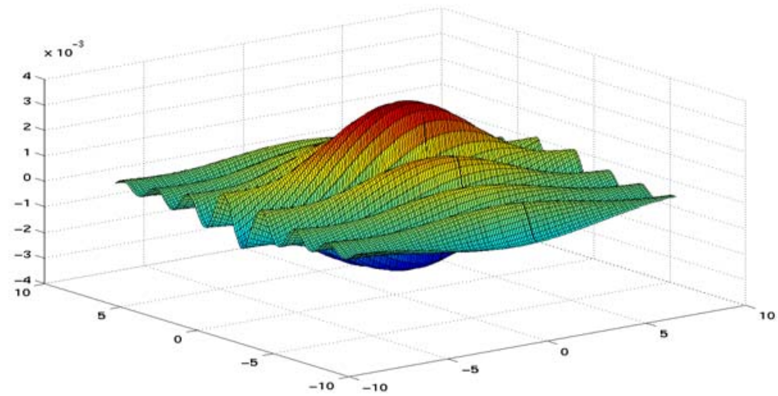

Figure 8: The extrapolated functions in plane section x = 3, for $\delta = 1$.

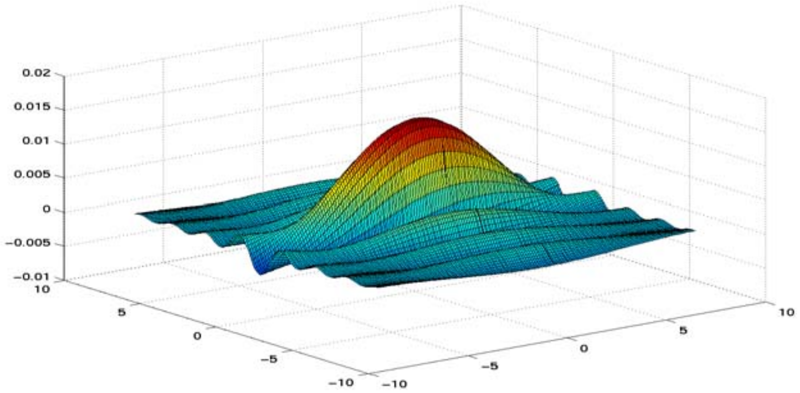

Figure 9: The extrapolated functions in plane section x = 3, for $\delta = 0.5$.

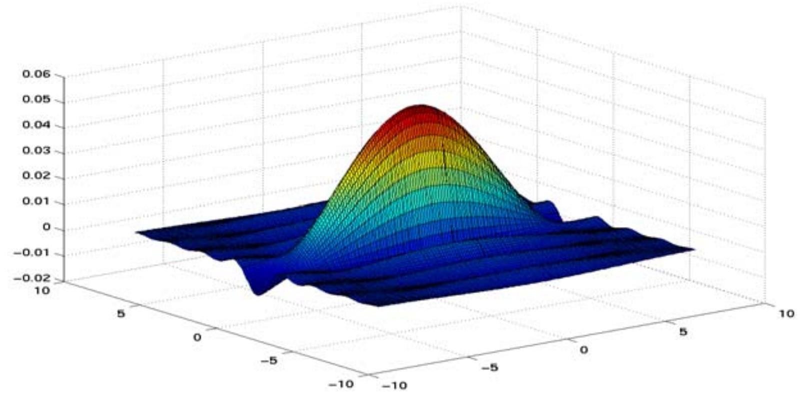

Figure 10: The extrapolated functions in plane section x = 3, for $\delta = 0.2$.

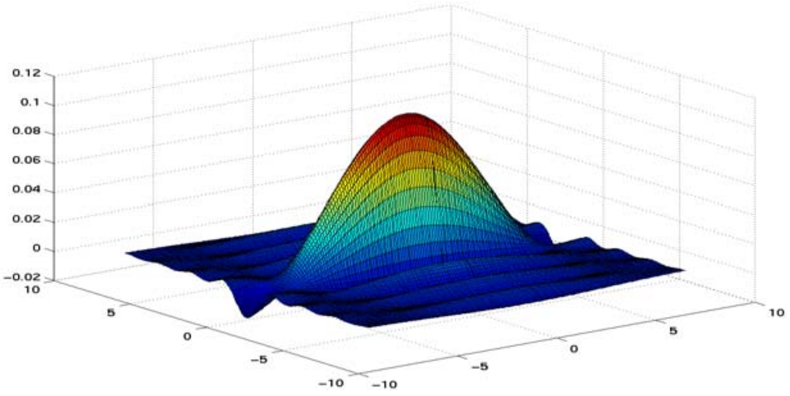

Figure 11: The extrapolated functions in plane section x = 3, for $\delta = 0.01$.

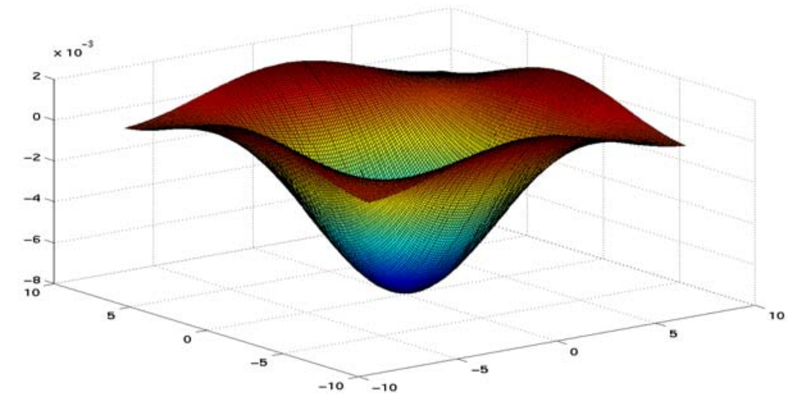

Figure 12: The original function in plane section y = -2.

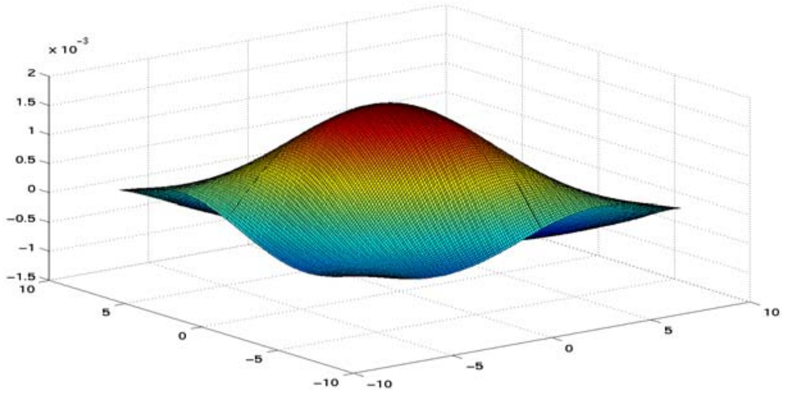

Figure 13: The extrapolated function in plane section: y = -2, for $\delta = 1$.

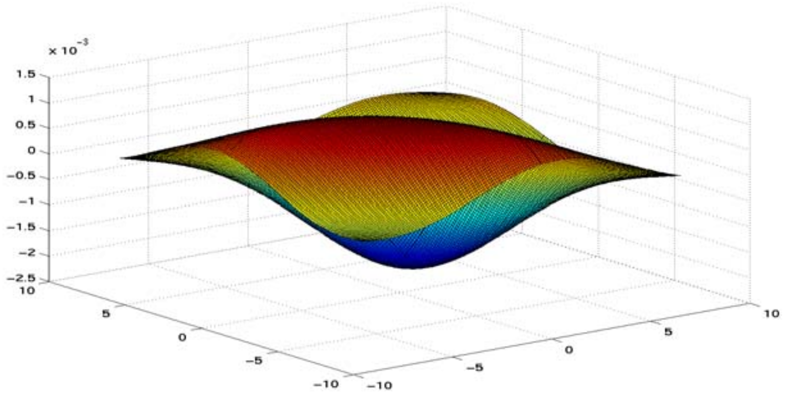

Figure 14: The extrapolated function in plane section: y = -2, for $\delta = 0.5$.

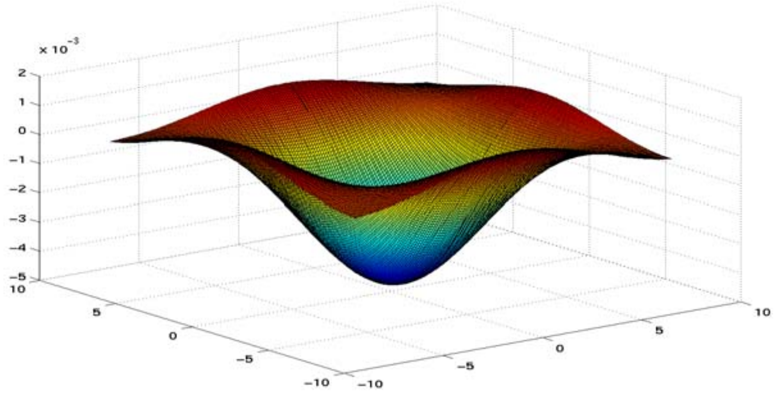

Figure 15: The extrapolated function in plane section: y = -2, for $\delta = 0.2$.

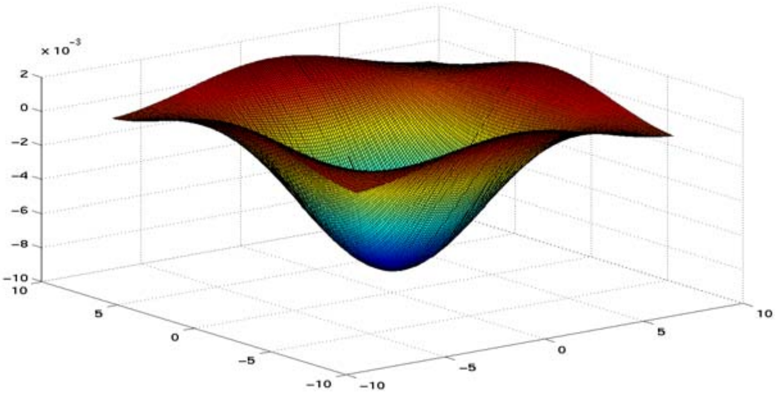

Figure 16: The extrapolated function in plane section: y = -2, for $\delta = 0.01$.

References

[1] A-L. Charbonniaud, J-F. Crouzet, and R. Gay, *Sur l'extrapolation des signoux d'energie finie a band limitee*, Rendiconti dell'Istituto di Matematica dell'Università di Trieste **28**, 145-182 (1998).

[2] A. Papoulis, *A new algorithm in spectral analysis and. bandlimited signal extrapolation*, IEEE Trans. Circuits. Syst. **CAS-22**, 735-742 (1975).

[3] R.W. Gershberg, *Superresolution through error energy reduction*, Optica Acta **21**, 709-720 (1974).

[4] L. Aizenberg, *Carleman's Formulas in Complex Analysis, Theory and Applications*, Mathematics and Its Applications, vol. **244** (Kluwer, 1993).