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Local geometric characterization of generalized quasiconformal mappings

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Abstract

One of characteristic properties of quasiconformal mappings is the quasiinvariance of n-module. The quasiinvariance is crucial in various applications. We provide here general conditions which are formulated in terms of the inequalities for certain set functions.

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1 Introduction

In this paper we continue to investigate the classes of mappings with finite mean dilatations. Our main goal is to generalize certain important classes of mappings: quasiconformal, quasiconformal in mean, etc.

The theory quasiconformal mappings in the plane has been appeared on the end of 1920-th in the works of Grötzsch and Lavrent'ev and is now a far developed area of geometric function theory with extremely reach applications.

A concept of quasiconformality in $\mathbb{R}^n$ was introduced by Lavrent'ev in 1938 as a suitable tool to construct some mathematical models of certain hydrodynamic problems. A point is that in multidimensional case the conditions of comformality are very rigid and therefore the class of conformal mappings in $\mathbb{R}^n, n > 2$, is narrow. Namely, by the fundamental Liouville theorem (1850) the only conformal mappings are the Möbius mappings, that is the finite compositions of reflections in spheres.

One of characteristic properties of quasiconformal mappings is a quasiinvariance of n-module of families of joining curves (or conformal capacity). It states that the module of a curve family is changed under a K-quasiconformal mapping only up to a factor at most K. All other properties of K-quasiconformal mappings can be derived from this inequality for K-quasiconformality (see, e.g., [1]). We shall consider more general inequalities than the quasiinvariance of n-moduli.

An essential deficiency of the moduli methods is that the exact values of moduli were found only for simple types of domains. In general case, one only can estimate these moduli, and in the most cases these estimates are not sharp. Moreover, these estimates do not exist for all values of parameter p of p-module.

Another approach to investigation of generalized quasiconformal mappings involves some local geometric characteristics which are based on appropriate change of the radii of the normal neighborhood systems. We discuss also the equivalence of the above methods. On other geometric methods, we refer to the survey paper of Srebro [2].

2 Quasiconformal dilatations

Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection. The numbers

$$H_I(A) = \frac{|detA|}{l^n(A)}, \quad H_O(A) = \frac{L^n(A)}{|detA|}, \quad H(A) = \frac{L(A)}{l(A)},$$

are called the *inner*, the *outer* and the *linear* dilatations of A, respectively. Here

$$l(A) = \min_{|h|=1} |Ah|, \quad L(A) = \max_{|h|=1} |Ah|,$$

and $detA$ is the determinant of A (see, e.g., [3]).

Obviously, all three dilatations are not less than 1. They have the following geometric interpretation. The image of the unit ball B^n under A is an ellipsoid $E(A)$. Let $B_I(A)$ and $B_O(A)$ be the inscribed and the circumscribed balls of $E(A)$, respectively.

Then

$$H_I(A) = \frac{mE(A)}{mB_I(A)}, \quad H_O(A) = \frac{mB_O(A)}{mE(A)},$$

and $H(A)$ is the ratio of the greatest and the smallest semi-axis of $E(A)$. Here $mA = m_n A$ denotes the n-dimensional Lebesgue measure of a set A.

Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the semi-axes of $E(A)$. Then

$$L(A) = \lambda_1, \quad l(A) = \lambda_n, \quad |detA| = \lambda_1 \cdot \ldots \cdot \lambda_n,$$

and we can also write

$$H_I(A) = \frac{\lambda_1 \cdot \ldots \cdot \lambda_{n-1}}{\lambda_n^{n-1}}, \quad H_O(A) = \frac{\lambda_1^{n-1}}{\lambda_2 \cdot \ldots \cdot \lambda_n}, \quad H(A) = \frac{\lambda_1}{\lambda_n}.$$

If $n = 2$ then $H_I(A) = H_O(A) = H(A)$. In the general case, we have the relations:

$$\begin{aligned} H(A) &\leq \min(H_I(A), H_O(A)) \leq H^{n/2}(A) \\ &\leq \max(H_I(A), H_O(A)) \leq H^{n-1}(A). \end{aligned} \tag{1}$$

Let G and G^* be two bounded domains in $\mathbb{R}^n$, $n \geq 2$, and let a mapping $f: G \to G^*$ be differentiable at a point $x \in G$. This means there exists a linear mapping $f'(x): \mathbb{R}^n \to \mathbb{R}^n$, called the (strong) derivative of the mapping f at x, such that

$$f(x + h) = f(x) + f'(x)h + \omega(x, h)|h|,$$

where $\omega(x, h) \to 0$ as $h \to 0$.

We denote

$$H_I(x, f) = H_I(f'(x)), \quad H_O(x, f) = H_O(f'(x)),$$

and

$$L(x, f) = L(f'(x)), \quad l(x, f) = l(f'(x)), \quad J(x, f) = det(f'(x)).$$

Proposition 1. *Let $f: G \to G^*$ be a K-quasiconformal homeomorphism, $1 \leq K < \infty$. Then*

(i) f is ACL (absolutely continuous on lines);

(ii) $f \in W^1_{n,loc}(G)$ (Sobolev class);

(iii) for almost every $x \in G$,

$$H_I(x, f) \leq K, \quad H_O(x, f) \leq K.$$

3 Global characteristics of quasiconformality

Now we define the quasiconformality of homeomorphisms in other terms (geometric or modular). Let $\mathcal{S}_k$ be a family of k-dimensional surfaces $\mathcal{S}$ in $\mathbb{R}^n$, $1 \leq k \leq n-1$ (curves for $k=1$). $\mathcal{S}$ is a k-dimensional surface if $\mathcal{S}:\ D_s \to \mathbb{R}^n$ is a homeomorphic image of the closed domain $D_s \subset \mathbb{R}^k$.

The p-module of $\mathcal{S}_k$ is defined by

$$M_p(\mathcal{S}_k) = \inf \int\limits_{\mathbb{R}^n} \rho^p \, dx, \quad p \geq 1,$$

where the infimum is taken over all Borel measurable functions $\rho \geq 0$ and such that

$$\int S\rho^k \, d\sigma_k \geq 1$$

for every $\mathcal{S} \in \mathcal{S}_k$. We call each such ρ to be an *admissible function* for $\mathcal{S}_k$.

The following proposition characterizes K-quasiconformality in the terms of n-moduli of $\mathcal{S}_k$ (see, e.g., [4], cf. [5]).

Proposition 2. *A homeomorphism f of a domain $G \subset \overline{\mathbb{R}^n}$ is K-quasiconformal, $1 \leq K < \infty$, if for any family $\mathcal{S}_k$, $1 \leq k \leq n-1$, of k-dimensional surfaces in G the double inequality*

$$K^{\frac{k-n}{n-1}} M_n(\mathcal{S}_k) \leq M_n(f(\mathcal{S}_k)) \leq K^{\frac{n-k}{n-1}} M_n(\mathcal{S}_k),$$

holds.

For more details about geometric definitions of quasiconformality see also [2].

A ring domain $D \subset \mathbb{R}^n$ is a finite domain whose complement consists of two components C_0 and C_1. Setting $F_0 = \partial C_0$ and $F_1 = \partial C_1$, we obtain two boundary components of D. For definiteness, let us assume that $\infty \in C_1$.

We say that a curve γ which *joins the boundary components* in D, if γ lies in D excluding its endpoints, one of which lies on F_0 and the second on F_1. A compact set Σ is said to *separate the boundary components* of D if $\Sigma \subset D$ and if C_0 and C_1 are located in different components of the complement $C\Sigma$ of Σ. Denote by Γ_D the family of all locally rectifiable curves γ, which join the boundary components of D, and by Σ_D the family of all compact piecewise smooth $(n-1)$-dimensional surfaces Σ, which separate the boundary components of D. For each quantity V associated with D such as subset of D or a family of sets contained in D, we let V^* denote its image under f.

Theorem 1. *A homeomorphism $f: G \to G^*$ is K-quasiconformal ($1 \le K < \infty$) if and only if there exists a constant K, $1 \le K < \infty$, such that for any ring domain $D \subset G$ either the inequalities*

$$M_p^n(\Sigma_D^*) \le K^{\frac{p}{n-1}} \big(mD^*\big)^{n-p} M_n^p(\Sigma_D),$$

$$M_q^n(\Sigma_D) \le K^{\frac{q}{n-1}} \big(mD\big)^{n-q} M_n^q(\Sigma_D^*),$$

hold for $n-1 < p < q \le n$ or inequalities

$$M_n^p(\Sigma_D^*) \le K^{\frac{p}{n-1}} \big(mD\big)^{p-n} M_p^n(\Sigma_D),$$

$$M_n^q(\Sigma_D) \le K^{\frac{q}{n-1}} \big(mD^*\big)^{q-n} M_q^n(\Sigma_D^*),$$

hold for $n \le p < q < (n-1)^2/(n-2)$.

The proof of this theorem is given in [6].

4 Gehring's characterization of quasiconformality

In 1973 Gehring proved the following important result

Gehring's Lemma. *Let G be an open subset of $\mathbb{R}^n$ and let $1 < p < \infty$. Suppose a nonnegative function h on G satisfies*

$$\fint_Q h^p \le A_p \Big(\fint_Q h\Big)^p$$

for all cubes $Q \subset G$ with a constant A_p independent of the cube. Then there exist a new exponent $s > p$ and a constant A_s depending only on p, n and A_p such that

$$\fint_Q h^s \le A_s \Big(\fint_Q h\Big)^s.$$

Here the symbol $\fint_Q h$ stands for the L^1-mean of h over the cube Q,

$$\fint_Q h = \frac{1}{mQ} \int_Q h.$$

In the same paper [7] Gehring derived the inequalities related to L^1 and L^n-means of the differential of K-quasiconformal mappings. He proved the following result (see, also [8]).

Integrability Theorem. *Every K-quasiconformal mapping $f: G \to \mathbb{R}^n$ belongs to the Sobolev space $W^1_{p,loc}$ with an exponent $p = p(n, K)$ greater than the dimension.*

This theorem not only extends an earlier result of Bojarski [9] for the planar case but also gives a new characteristic property for K-quasiconformal mappings.

An essential complement to Gehring's result is given by Reshetnyak [10].

5 Generalized quasiconformal mappings

The classes of mappings quasiconformal in the mean are studied more than 40 years (see, e.g., [11]). One of recent developments in this field is provided in [12].

Consider the quantities

$$H_{I,\alpha}(A) = \frac{|J(A)|}{l^\alpha(A)}, \quad H_{O,\alpha}(A) = \frac{L^\alpha(A)}{|J(A)|}, \quad \alpha \geq 1.$$

Such dilatations were applied in [6, 13-17]. For $\alpha = n$, the values of $H_{I,\alpha}(A)$ and $H_{O,\alpha}(A)$ coincide with $H_I(A)$ and $H_O(A)$, respectively.

We consider the homeomorphisms f which are differentiable almost everywhere in G and fix the real numbers α, β satisfying $1 \leq \alpha < \beta < \infty$. Put

$$HI_{\alpha,\beta}(f) = \int\limits_G H_{I,\alpha}^{\frac{\beta}{\beta-\alpha}}(x, f)\, dx, \quad HO_{\alpha,\beta}(f) = \int\limits_G H_{O,\beta}^{\frac{\alpha}{\beta-\alpha}}(x, f)\, dx,$$

where $H_{I,\alpha}(x, f) = H_{I,\alpha}(f'(x))$, $H_{O,\beta}(x, f) = H_{O,\beta}(f'(x))$. We call these values the *inner* and the *outer* mean dilatations of a mapping f of a given domain G.

The main purpose to introduce the inner and outer mean dilatations relies on the following theorem.

Theorem 2 ([18]). *Let $f: G \to G^*$ be a homeomorphism satisfying:*

(iv) f and f^{-1} are ACL;

(v) f and f^{-1} are differentiable a.e. in G and G^*, respectively;

(vi) the Jacobians $J(x, f)$ and $J(y, f^{-1})$ do not vanish a.e. in G and G^*, respectively.

Then for every fixed values $\alpha, \beta, \gamma, \delta$ such that $1 \leq \alpha < \beta < \infty$, $1 \leq \gamma < \delta < \infty$, and for any ring domain $D \subset G$ the inequalities

$$M_\alpha^\beta(\mathcal{S}_k^*) \leq HI_{\alpha,\beta}^{\beta-\alpha}(f) M_\beta^\alpha(\mathcal{S}_k),$$

$$M_{\gamma}^{\delta}(\mathcal{S}_k) \le HO_{\gamma,\delta}^{\delta-\gamma}(f) M_{\delta}^{\gamma}(\mathcal{S}_k^*),$$

hold, where $\mathcal{S}_k^* = f(\mathcal{S}_k)$.

Define for the fixed real numbers $\alpha, \beta, \gamma, \delta$ such that $1 \le \alpha < \beta < \infty$, $1 \le \gamma < \delta < \infty$, the *class* $\mathcal{B}(G)$ of such homeomorphisms $f:\ G \to G^*$ which satisfy:
(vii) f and f^{-1} are ACL-homeomorphisms,
(viii) f and f^{-1} are differentiable, with the Jacobians $J(x,f) \ne 0$ and $J(y,f^{-1}) \ne 0$ a.e. in G and G^*, respectively,
(ix) the inner and the outer mean dilatations $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$ are finite.

In other words, the class $\mathcal{B}(G)$ consists of *mappings with finite mean dilatations.* The case $\beta = \delta = n$ was considered in [19].

It follows that for any $f \in \mathcal{B}(G)$, we have the equalities

$$HI_{\alpha,\beta}(f^{-1}) = HO_{\alpha,\beta}(f), \quad HO_{\alpha,\beta}(f^{-1}) = HI_{\alpha,\beta}(f).$$

The relations (1) show that in the classical case of quasiconformal mappings, their dilatations are finite or infinity simultaneously. However, this is not true for the mean dilatations. The following example shows that the unboundedness of one of dilatations does not depend on the value of another mean dilatation.

Example 1. Consider the unit cube

$$G = \{x = (x_1, \dots, x_n) : 0 < x_k < 1, k = 1, \dots, n\},$$

and let

$$f(x) = \left(x_1, \dots, x_{n-1}, \frac{x_n^{1-c}}{1-c}\right), \quad 0 < c < 1.$$

The calculation gives

$$HI_{\alpha,\beta}(f) = \int\limits_G H_{I,\alpha}^{\frac{\beta}{\beta-\alpha}}(x,f)\,dx = \int\limits_0^1 x_n^{-\frac{c\beta}{\beta-\alpha}}\,dx_n,$$

$$HO_{\gamma,\delta}(f) = \int\limits_G H_{O,\delta}^{\frac{\gamma}{\delta-\gamma}}(x,f)\,dx = \int\limits_0^1 x_n^{-\frac{c(\delta-1)\gamma}{\delta-\gamma}}\,dx_n,$$

which yields

$$HI_{\alpha,\beta}(f) < \infty \iff 0 < c < 1 - \alpha/\beta,$$

$$HI_{\alpha,\beta}(f) = \infty \iff 1 - \alpha/\beta \le c < 1,$$

$$HO_{\gamma,\delta}(f) < \infty \iff 0 < c < 1 - (\gamma - 1)\delta/(\delta - 1)\gamma,$$

$$HO_{\gamma,\delta}(f) = \infty \iff 1 - (\gamma - 1)\delta/(\delta - 1)\gamma \le c < 1.$$

This allows us to choose the parameters $c, \alpha, \beta, \gamma, \delta$ so that one obtains the desired relations between $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$.

6 Relationship between classes with finite mean dilatations

The following theorems describe the relationship between the classes $\mathcal{B}(G)$.

Theorem 3. *Suppose that $\alpha, \beta, \gamma, \delta$ are fixed real numbers such that $n-1 \le \alpha < \beta < \infty$, $n - 1 \le \gamma < \delta < \infty$. Then the mappings of $\mathcal{B}(G)$ belong to the Sobolev classes $W^1_{p,loc}(G)$ and $W^1_{q,loc}(G^*)$ with $p = \max\bigl(\gamma, \beta/(\beta - n + 1)\bigr)$ and $q = \max\bigl(\alpha, \delta/(\delta - n + 1)\bigr)$.*

The proof of this theorem is given in [6].

Theorem 4. *Let $\alpha, \beta, \gamma, \delta, r, s, t, u$ be fixed real numbers such that $1 \le r < \alpha < \beta < s < \infty$ and $1 \le t < \gamma < \delta < u < \infty$. Then*

(a) $\mathcal{B}(G, G^*, \alpha, \beta, \gamma, \delta) \subset \mathcal{B}(G, G^*, r, \beta, \gamma, \delta)$,

(b) $\mathcal{B}(G, G^*, \alpha, \beta, \gamma, \delta) \subset \mathcal{B}(G, G^*, \alpha, s, \gamma, \delta)$,

(c) $\mathcal{B}(G, G^*, \alpha, \beta, \gamma, \delta) \subset \mathcal{B}(G, G^*, \alpha, \beta, t, \delta)$,

(d) $\mathcal{B}(G, G^*, \alpha, \beta, \gamma, \delta) \subset \mathcal{B}(G, G^*, \alpha, \beta, \gamma, u)$.

Example 2. Let $n \le \alpha < \beta < \infty$ and $n \le \gamma < \delta < \infty$. Fix two numbers $0 < a < 1$ and $p > 0$ and consider two spherical systems of coordinates (r, φ_i) and (ρ, ψ_i) on the n-dimensional balls $B(0, a)$ and $B\bigl(0, a^{\frac{\gamma-n}{\gamma}} \ln^{-p} \frac{1}{a}\bigr)$ respectively. Here $B(x, h) = \{y \in \mathbb{R}^n : |y - x| < h\}$ denotes n-dimensional ball of radius h centered at x, $\Omega_n = mB(0, 1)$, $\omega_{n-1} = m_{n-1}B(0, 1)$

It is easy to verify that the mapping

$$g = \Big\{ \rho = r^{\frac{\gamma - n}{\gamma}} \ln^{-p} \frac{1}{r}, 0 < r < a < 1, \psi_i = \varphi_i, 0 \le \varphi_i < \pi, i = 1, \dots, n-2,$$
$$0 \le \varphi_{n-1} < 2\pi, \rho(0) = 0 \Big\}$$

moves the ball $B(0,a)$ to $B\big(0, a^{\frac{\gamma-n}{\gamma}} \ln^{-p} \frac{1}{a}\big)$. Therefore, g and g^{-1} are differentiable a.e. with nonzero Jacobians in $B(0,a)$ and $B\big(0, a^{\frac{\gamma-n}{\gamma}} \ln^{-p} \frac{1}{a}\big)$, respectively, which yields that the value

$$\begin{aligned} HO_{\gamma,\delta}(g) &= \int\limits_{B(0,a)} \left(\frac{L^\delta(x,g)}{|J(x,g)|} \right)^{\frac{\gamma}{\delta-\gamma}} dx \\ &= \omega_{n-1} \int\limits_0^a r^{\frac{n(n-\gamma)}{\delta-\gamma} - 1} \ln^{-\frac{p\gamma(\delta-n)}{\delta-\gamma}} \frac{1}{r} \left(\frac{\gamma - n}{\gamma} + \ln^{-1} \frac{1}{r} \right)^{\frac{\gamma}{\delta-\gamma}} dr \end{aligned}$$

is finite only when

$$n = \gamma \quad \text{and} \quad 0 < p \le \frac{\delta}{n(\delta - n)},$$

and $HO_{\gamma,\delta}(g) = \infty$ for $n < \gamma$. Moreover,

$$\int\limits_{B(0,a)} L^\gamma(x,g) dx = \omega_{n-1} \int\limits_0^a r^{-1} \ln^{-p\gamma} \frac{1}{r} \left(\frac{\gamma - n}{\gamma} + \ln^{-1} \frac{1}{r} \right)^\gamma dr < \infty$$

under $p > 1/\gamma$, which shows that this mapping, being in the Sobolev space $W^1_{\gamma,loc}$ does not belong to our class $\mathcal{B}(G)$.

Hence, for the indicated restrictions to parameters $\alpha, \beta, \gamma, \delta$ the class $\mathcal{B}(G)$ is a proper subset of $W^1_{\gamma,loc}(G)$.

On the other hand,

$$\int\limits_{B(0,a)} L^{\gamma+\varepsilon}(x,g) dx = \omega_{n-1} \int\limits_0^a r^{-1-\frac{n\varepsilon}{\gamma}} \ln^{-p(\gamma+\varepsilon)} \frac{1}{r} \left(\frac{\gamma - n}{\gamma} + \ln^{-1} \frac{1}{r} \right)^{\gamma+\varepsilon} dr = \infty$$

for any positive ε and p. Thus the mapping g belongs to the class $\mathcal{B}(G)$ for

$$\frac{1}{n} \le p \le \frac{\delta}{n(\delta - n)}$$

and $W^1_{n,loc}(G)$, but g does not belong for $W^1_{n+\varepsilon,loc}(G)$. We conclude that the Integrability Theorem does not hold for the class of mappings with finite mean dilatations.

7 Global characterization of generalized quasiconformal mappings

We now establish the inequalities which generalize the quasiinvariance of module in the case of homeomorphisms with finite mean dilatations and derive some differential and geometric properties of such mappings.

To this end, we consider also certain set functions.

Let Φ be a finite nonnegative function in domain G defined for open subsets E of G so that

$$\sum_{k=1}^{m} \Phi(E_k) \leqslant \Phi(E)$$

for any finite collection $\{E_k\}_{k=1}^m$ of nonintersecting open sets $E_k \subset E$. We denote the class of such set functions Φ by $\mathcal{F}$.

Now we introduce some new classes of homeomorphisms depending on the values of parameters $\alpha, \beta, \gamma, \delta$ and on the set functions.

Fix the numbers $\alpha, \beta, \gamma, \delta$ which satisfy

$$n-1 \le \alpha < \beta < \infty, \quad n-1 \le \gamma < \delta < \infty,$$

and assume that there exists a nonempty family of homeomorphisms $f : G \to G^*$ such that there are two set functions $\Phi, \Psi \in \mathcal{F}$ not depending from f so that for each ring domain $D \subset G$ the inequalities

$$M_\alpha^\beta(\Sigma_D^*) \le \Phi^{\beta-\alpha}(D) M_\beta^\alpha(\Sigma_D), \tag{2}$$

$$M_\gamma^\delta(\Sigma_D) \le \Psi^{\delta-\gamma}(D) M_\delta^\gamma(\Sigma_D^*), \tag{3}$$

hold.

The class of such homeomorphisms will be denoted by $\mathcal{MS}(G)$. (In fact, it depends also on $\alpha, \beta, \gamma, \delta$.) Their main properties of this class are given by following

Theorem 5 ([18]). *Let* $n-1 < \alpha < \beta \le n$ *and* $n-1 < \gamma < \delta \le n$ *or*

$$n \le \alpha < \beta < \frac{(n-1)^2}{n-2} \quad \textit{and} \quad n \le \gamma < \delta < \frac{(n-1)^2}{n-2}.$$

Then every mapping $f \in \mathcal{MS}(G)$ admits the following properties:

(a′) f is ACL in G;

(b′) f^{-1} is ACL in G^*;

(c′) $f \in W^1_{a,loc}(G)$ with $a = \max\Big(\gamma, \beta/(\beta - n + 1)\Big)$;

(d′) $f^{-1} \in W^1_{b,loc}(G^*)$ with $b = \max\Big(\alpha, \delta/(\delta - n + 1)\Big)$.

Remarks.

1. It is enough only one inequality ((2) or (3)) to provide (a') and (b').
2. f and f^{-1} both possess N-property (see, e.g., [20]).
3. To characterize a quasiconformal mapping again enough only one inequality ((2) or (3)) with $\alpha = \beta = n$ or $\gamma = \delta = n$.
4. Another characterization of the class $\mathcal{MS}(G)$ can be given in the terms of the moduli of Γ_D instead of moduli of Σ_D.

8 Local characterization of generalized quasiconformal mappings

Let x be an arbitrary point in $\mathbb{R}^n$. Assume that some closed neighborhood $\mathcal{G}_t(x)$ of x is defined for any $t \in (0, 1]$. We say that a set of the neighborhoods $\mathcal{G}_t(x)$ of the point x constitutes a *normal system*, if there exists a continuous function $v : \ R^n \to \mathbb{R}$ such that $v(x) = 0$, $v(y) > 0$ for any $y \neq x$. Here $\mathcal{G}_t(x) = \{y \in \mathbb{R}^n : \ v(y) \leq t\}$ for any $t \in (0, 1]$. Let $\Gamma_t(x) = \{y \in \mathbb{R}^n : v(y) = t\}$ denote the boundary of $\mathcal{G}_t(x)$.

The function v is called the *generating function* for a given normal system $\{\mathcal{G}_t(x)\}$.

Denote

$$r(x,t) = \inf_{y \in \Gamma_t(x)} |y - x|, \quad \mathcal{R}(x,t) = \sup_{y \in \Gamma_t(x)} |y - x|.$$

These values $r(x,t)$ and $\mathcal{R}(x,t)$ are equal, respectively, to the minimal and the maximal radii of the neighborhood $\mathcal{G}_t(x)$. The limit

$$\Delta(x) = \limsup_{t \to 0} \frac{\mathcal{R}(x,t)}{r(x,t)}$$

is called the *regularity parameter* of the family $\{\mathcal{G}_t(x), 0 < t \leq 1\}$. Any such system $\{\mathcal{G}_t(x)\}$ is called the *regular normal system*, provided $\Delta(x) < \infty$.

Let $f: \ G \to G^*$ be a homeomorphism and let $\{\mathcal{G}_t(x)\}$ be a normal system of neighborhoods of $x \in G$. One can introduce similar to above the minimal and the maximal radii for the image of $\mathcal{G}_t(x)$ by

$$r^*(x,t) = \inf_{y \in \Gamma_t(x)} |f(y) - f(x)|, \quad \mathcal{R}^*(x,t) = \sup_{y \in \Gamma_t(x)} |f(y) - f(x)|$$

and

$$\Delta^*(x) = \limsup_{t \to 0} \frac{\mathcal{R}^*(x,t)}{r^*(x,t)}.$$

Yu.G. Reshetnyak [10] has investigated quasiconformal mappings of the space domains using the radii of the normal regular system of neighborhoods. He called a mapping f quasiconformal at a point $x \in G$ if there exists a normal regular system $\{\mathcal{G}_t(x)\}$ of neighborhoods of x such that $\Delta(x)\Delta^*(x) < \infty$.

The *upper* and *lower* derivatives of a set function $\Phi \in \mathcal{F}$ at a point $x \in G$ are defined by

$$\overline{\Phi'}(x) = \lim_{h \to 0} \sup_{d(Q) < h} \frac{\Phi(Q)}{mQ}, \qquad \underline{\Phi'}(x) = \lim_{h \to 0} \inf_{d(Q) < h} \frac{\Phi(Q)}{mQ},$$

where Q ranges over all open cubes and open balls such that $x \in Q \subset G$ and $d(Q) = diamQ$. Due to [21], these derivatives have the following properties:

(x) $\overline{\Phi'}(x)$ and $\underline{\Phi'}(x)$ are Borel's functions;

(xi) $\overline{\Phi'}(x) = \underline{\Phi'}(x) < \infty$ a.e. in G;

(xii) for each open set $V \subset G$,

$$\int\limits_V \overline{\Phi'}(x)\, dx \le \Phi(V).$$

Using these set functions, we define for the fixed real numbers $\alpha, \beta, \gamma, \delta$ such that $1 \le \alpha < \beta < \infty$ and $1 \le \gamma < \delta < \infty$, the class $\mathcal{H}(G)$ of homeomorphisms $f: G \to G^*$ which satisfy:

(xiii) there exist $\Phi, \Psi \in \mathcal{F}$ in G,

(xiv) for any point $x \in G$ there exists $\{\mathcal{G}_t(x)\} \subset G$,

(xv) the inequalities

$$\limsup_{t \to 0} \frac{mf(B(x, \mathcal{R}(x,t)))\mathcal{R}^{\alpha - n}(x,t)}{\Omega_n r^{*\alpha}(x,t)} \le \left[\Phi'(x)\right]^{\frac{\beta - \alpha}{\beta}}, \tag{4}$$

$$\limsup_{t \to 0} \frac{\Omega_n \mathcal{R}^{*\delta}(x,t)}{mf(B(x, r(x,t)))r^{\delta - n}(x,t)} \le \left[\Psi'(x)\right]^{\frac{\delta - \gamma}{\gamma}}. \tag{5}$$

hold for all points $x \in G$ at which the derivatives $\Phi'(x)$ and $\Psi'(x)$ exist.

9 Equivalence of analytic and geometric descriptions

We now are enable to establish that classes $\mathcal{B}(G)$ and $\mathcal{H}(G)$ are coincide. The proof of this fact is accomplished in several steps and relies on an idea of the classical Menshoff paper [22]. (See also [10]).

Lemma 1. *Let $f \in \mathcal{H}(G)$, then f is ACL-mapping and is differentiable a.e. in G.*

Sketch of the proof. First, we show that f is ACL-mapping in G. Fix for each $x \in G$ a normal regular system $\{\mathcal{G}_t(x)\}$ of neighborhoods such that $\mathcal{G}_t(x) \subset G$ for any $t \in (0,1]$. Consider an arbitrary point $a = (a_1, \dots, a_n) \in G$ and $h > 0$ so that the cube $\bar{Q} = \bar{Q}(a,h)$ belongs to G, where $Q(a,h) = \{x : |x_i - a_i| < h, i = 1, \dots, n\}$. Denote by C_k, $k = 1, \dots, n$, the intersection of $\bar{Q}$ with the plane $P_k(a) = \{x \in \mathbb{R}^n : x_k = a_k\}$, and let $p(x)$ be the segment $|x_k - a_k| \leq h/2$ of the line passing through x ($x \in C_k$) parallel to the kth coordinate axis x_k. Similarly, let $p(A)$ denote the union of all segments $p(x)$, when $A \subset C_k$ and $x \in A$.

We show that for almost all $x \in C_k$ (with respect to $(n-1)$-dimensional Lebesgue measure) the restriction of f to $p(x)$ admits the Lusin N-property.

Further, for $x, \tilde{x} \in G, \tilde{x} \neq x$, we define

$$k(x) = \limsup_{\tilde{x} \to x} \frac{|f(\tilde{x}) - f(x)|}{|\tilde{x} - x|}.$$

Then $k(x) < \infty$ for almost all $x \in G$, and for any open set $E \subset G$ we have

$$\int\limits_E k^{\frac{\beta}{\beta-n+1}}(x)\,dx \leq \Big[\Phi(E)\Big]^{\frac{\beta-\alpha}{\beta-n+1}} \Big[mf(E)\Big]^{\frac{\alpha-n+1}{\beta-n+1}} < \infty,$$

$$\int\limits_E k^{\gamma}(x)\,dx \leq \Big[\Phi(E)\Big]^{\frac{\delta-\gamma}{\delta}} \Big[mf(E)\Big]^{\frac{\gamma}{\delta}} < \infty.$$

These inequalities allow us to the classical Stepanov theorem [23] on differentiability and obtain that f is differentiable almost everywhere in G.

Lemma 2. *Let $f \in \mathcal{H}(G)$, then $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$ are finite.*

Lemma 3. *If $f \in \mathcal{B}(G)$, then $f \in \mathcal{H}(G)$.*

The proofs of Lemmas 2 and 3 are similar to the corresponding lemmas from [16]. Now we can now formulate the following theorem which is the main result of this section.

Theorem 6. *The classes $\mathcal{B}(G)$ and $\mathcal{H}(G)$ coincide.*

We now mention about some special cases. The first of them is the standard quasiconformality in $\mathbb{R}^n$. One can obtain the class of usual quasiconformal mappings not only when $\alpha = \beta = \gamma = \delta = n$, but also if the following conditions hold:
(xvi) only one of pairs α, γ or β, δ is equal to n;
(xvii) the set functions of D are reduced to the product $CmesE$, where $mesE$ denote the Lebesgue n-measure on D or D^*.

We set

$$K_I(x, \{\mathcal{G}_t(x)\}) = \limsup_{t\to 0} \frac{mf(B(x, \mathcal{R}(x,t)))}{\Omega_n r^{*n}(x,t)},$$

and

$$K_O(x, \{\mathcal{G}_t(x)\}) = \limsup_{t\to 0} \frac{\Omega_n \mathcal{R}^{*n}(x,t)}{mf(B(x, r(x,t)))}.$$

Theorem 7. *A homeomorphism $f: G \to G^*$ is quasiconformal in the domain G if and only if for almost all $x \in G$ there exist the normal regular systems $\{\mathcal{G}_t(x)\} \subset G$ of neighborhoods of x which satisfy*

$$K_I(x, \{\mathcal{G}_t(x)\}) \leq K(\{\mathcal{G}_t\}) < \infty, \quad K_O(x, \{\mathcal{G}_t(x)\}) \leq K(\{\mathcal{G}_t\}) < \infty.$$

Put now

$$K = \inf K(\{\mathcal{G}_t\}),$$

where the infinum is taken over all such systems of neighborhoods. This value is the *quasiconformality coefficient* of f in G.

In the planar case, $n = 2$, we obtain instead of (4),(5) the following bounds

$$\limsup_{t\to 0} \frac{\mathcal{R}^*(x,t)}{r(x,t)} \left(\frac{\mathcal{R}(x,t)}{r^*(x,t)}\right)^{\alpha-1} \leq \left[\Phi'(x)\right]^{\frac{\beta-\alpha}{\beta}},$$

$$\limsup_{t\to 0} \frac{\mathcal{R}(x,t)}{r^*(x,t)} \left(\frac{\mathcal{R}^*(x,t)}{r(x,t)}\right)^{\delta-1} \leq \left[\Psi'(x)\right]^{\frac{\delta-\gamma}{\gamma}}.$$

In other words, we can characterize geometrically in terms of radii of normal neighborhood systems many well-known classes of mappings (quasiconformal, quasiconformal in the mean, enc).

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