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Convergences almost everywhere and locally almost everywhere in $*$-algebras of locally measurable operators

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Abstract

In this paper, we consider $*$-algebras $LS(M)$ of locally measurable operators affiliated to a von Neumann algebra M, and study different kinds of convergences in this algebras, the convergence almost everywhere and the convergence locally almost everywhere. We also study a relationship between these two convergences.

Introduction

One of the first approaches to introduce a "noncommutative version" of the ring of measurable functions was suggested by I. Segal [1], who considered a *$*$-algebra $S(M)$ of measurable operators affiliated to a von Neumann algebra M.* Later, for purposes of noncommutative integration, one considered the $*$-subalgebras of $S(M)$, $S(M, \tau)$, of all τ-measurable operators associated with a faithful normal semi-finite trace τ on M, see, e.g., [2–4]. The algebras $S(M, \tau)$ and $S(M)$ are $*$-algebras of closed densely defined linear operators that act on a Hilbert space H the same for the von Neumann algebra M itself. In such a case, all these operators are affiliated to M and the algebraic operations for these $*$-algebras coincide with the operation of the "strong sum", the "strong product", passing to the adjoint, and the usual multiplication by scalars. The von Neumann algebra M is a $*$-subalgebra of $S(M, \tau)$ and $S(M)$, and coincides with the set of all bounded operators in $S(M, \tau)$ and $S(M)$. A more general class of $*$-algebras of closed operators that act on a Hilbert space H and that are affiliated to a von Neumann algebra M was introduced by Dixon in [5] who called them EW^*-algebras. In addition to the mentioned above $*$-algebras $S(M)$ and $S(M, \tau)$, *$*$-algebras $LS(M)$ of locally measurable operators affiliated to M* are also EW^*-algebras [6, 7]. B.S. Zakirov and V.I. Chilin have shown in [8] that any EW^*-algebra $\mathcal{A}$ such that $\mathcal{A}\bigcap\mathcal{B}(H) = M$, where $\mathcal{B}(H)$ is the algebra of all bounded linear operators acting on H, is a $*$-subalgebra of $LS(M)$. This explains uniqueness of the $*$-algebra $LS(M)$ for a von Neumann algebra M in the class of EW^*-algebras.

In this paper, we consider $*$-algebras $LS(M)$, study different types of convergence in these algebras, i.e., convergences almost everywhere and locally almost everywhere, and study a relationship between these two convergences.

We employ the terminology and notations used in the theory of von Neumann algebras [9, 10] and the theory of measurable operators [1, 3, 4, 7].

1 Preliminaries

Let H be a Hilbert space, $\mathcal{B}(H)$ the algebra of all bounded operators acting on H, M a von Neumann algebra in $\mathcal{B}(H)$, $P(M)$ the complete lattice of all orthogonal projections in M.

A linear space D in H is called *affiliated to* M, denoted by $D\,\eta\,M$, if $U(D) \subset D$ for any unitary operator U from the commutant

$$M' = \{S \in \mathcal{B}(H) : \ ST = TS \ \forall\ T \in M\}$$

of the von Neumann algebra M. If D is a closed subspace of H and P_D is an operator of the orthogonal projection onto D, then $D\,\eta\,M$ if and only if $P_D \in P(M)$.

A linear operator T that acts on a Hilbert space H and has domain $D(T)$ is called *affiliated to* M, denoted by $T\,\eta\,M$, if $U(D(T)) \subset D(T)$ for any unitary operator U in the commutant M' and $UT\xi = TU\xi$ for all $\xi \in D(T)$. It is clear that if $T \in \mathcal{B}(H)$ and $T\,\eta\,M$, then $T \in M$.

A closed linear operator T with domain $D(T) \subset H$ is called *measurable with respect to a von Neumann algebra* M [1], if $T\,\eta\,M$ and there exists a sequence of projections, $\{P_n\}_{n=1}^{\infty} \subset P(M)$, such that $P_n \uparrow I$, $P_n(H) \subset D(T)$, and $P_n^{\perp} = I - P_n$ is a finite projection in M for all $n = 1, 2, \ldots$, where I is the identity in the von Neumann algebra M.

Denote by $S(M)$ the set of all linear operators on H, measurable with respect to the von Neumann algebra M. If $T \in S(M)$, $\lambda \in \mathbf{C}$, where $\mathbf{C}$ is the field of complex numbers, then $\lambda T \in S(M)$ and the operator T^*, adjoint to T, is also measurable with respect to M [1]. Moreover, if $T, S \in S(M)$, then the operators $T + S$ and TS are defined on dense subspaces and admit closures that are called, correspondingly, the strong sum and the strong product of the operators T and S, and are denoted by $T \dot{+} S$ and $T * S$. It was shown in [1] that $T \dot{+} S$ and $T * S$ belong to $S(M)$ and these algebraic operations make $S(M)$ a $*$-algebra with the identity I over the field $\mathbf{C}$. Here, M is a $*$-subalgebra of $S(M)$. In what follows, the strong sum and the strong product of operators T and S will be denoted in the same way as the usual operations, by $T + S$ and TS.

If T is a closed linear operator with the domain dense in H and $T = U|T|$ is the polar decomposition of the operator T, where $|T| = (T^*T)^{\frac{1}{2}}$ is the absolute value of T and U is the corresponding partial isometry, then $T \in S(M)$ if and only if $U \in M$ and $|T| \in S(M)$ [7]. The following proposition gives a convenient criterion for a closed operator T to be measurable in terms of the spectral family for $|T|$.

Proposition 1 ([7]). *Let T be a closed operator on H, $T \,\eta\, M$, $T = U|T|$ the polar decomposition of T, $\{E_\lambda\}$ the spectral family of projections for $|T|$, $\lambda \in \boldsymbol{R}$, where $\boldsymbol{R}$ is the field of real numbers. Then $U \in M$ and $E_\lambda \in P(M)$ for all $\lambda \in \boldsymbol{R}$. Also, $T \in S(M)$ if and only if the domain $D(T)$ of the operator T is dense in H and $E_\lambda^{\perp}$ is a finite projection for some $\lambda > 0$.* □

To prove Proposition 1.1, one uses the following lemma in an essential way. This lemma will be used later.

Lemma 1 ([7]). *Let T be a closed operator on H with dense domain $D(T)$, $T \,\eta\, M$, and $\{E_\lambda\}$ be the spectral family of projections for $|T|$, $\lambda \in \boldsymbol{R}$. If $P \in P(M)$, $P(H) \subseteq D(T)$, $TP \in \mathcal{B}(H)$, and $\|TP\|_{\mathcal{B}(\mathcal{H})} < \lambda$, then $E_\lambda^{\perp} \precsim P^{\perp}$ (recall that the relation $E \precsim Q$ for projections $E, Q \in P(M)$ means that $E \sim E_1 \leqslant Q$, and the equivalence of projections, $E \sim E_1$, is equivalent to existence of a partial isometry $V \in M$ such that $V^*V = E_1$ and $VV^* = E$).* □

It directly follows from Proposition 1.1 that, in the case where M is a type III von Neumann algebra or M is a type I factor, we always have $S(M) = M$. For von Neumann algebras of type II, the latter identity is not true already. The proof of this fact is based on the following proposition.

Proposition 2 ([11]). *If there exists an increasing sequence of projections* $\{E_n\}$ *in* M *such that* $E = \sup_{n \geqslant 1} E_n$ *is a finite projection, and* $E_n \neq E$ *for all* $n = 1, 2, \ldots$, *then* $S(M) \neq M$. □

Corollary 1. *If* M *is a von Neumann algebra of type* II, *then* $S(M) \neq M$. □

The following proposition gives conditions that are necessary and sufficient for $*$-algebras $S(M)$ and M to coincide.

Proposition 3 ([11]). *The following statements are equivalent.*

(i) $S(M) = M$.

(ii) M *can be represented as a direct sum,* $M = \sum\limits_{n=0}^{m} M_n$, *where* M_0 *is a von Neumann algebra of type* III, *and* M_n *are factors of type* I, $n = 1, 2, \ldots, m$, *and* m *is a natural number (some terms could be omitted).*

□

A closed linear operator T acting on a Hilbert space H is called *locally measurable with respect to a von Neumann algebra* M if $T \eta M$ and there exists a sequence $\{Z_n\}_{n=1}^{\infty}$ of central projections in M such that $Z_n \uparrow I$ and $TZ_n \in S(M)$ for all $n = 1, 2, \ldots$ [7].

Denote by $LS(M)$ the set of all linear operators that are locally measurable with respect to M. It was proved in [7] that $LS(M)$ is a $*$-algebra over the field $\mathbf{C}$ with identity I, the operations of strong addition, strong multiplication, and passing to the adjoint (the multiplication by a scalar is defined as usual with the assumption $0 * T = 0$.) In such a case, $S(M)$ is a $*$-subalgebra in $LS(M)$. In the case where M is a finite von Neumann algebra or a factor, the algebras $S(M)$ and $LS(M)$ coincide. This is not true in the general case. The following proposition gives a sufficient condition for these algebras to be distinct.

Proposition 4 ([11]). *If a von Neumann algebra* M *contains a sequence* $\{Z_n\}_{n=1}^{\infty}$ *of central projections, increasing to the identity, such that* $(I - Z_n)$ *is not a finite projection,* $n = 1, 2, \ldots$, *then* $LS(M) \neq S(M)$. □

Proposition 1.4 gives at once the following.

Corollary 2. *If a von Neumann algebra* M *is a direct product of an infinite number of von Neumann algebras that are not finite, then* $LS(M) \neq S(M)$. □

The following proposition gives a criterion for the $*$-algebras $LS(M)$ and $S(M)$ to coincide.

Proposition 5 ([11]). *The following statements are equivalent.*

(i) $LS(M) = S(M)$.

(ii) *M can be represented as a direct sum, $M = \sum_{n=0}^{m} M_n$, where M_0 is a finite von Neumann algebra and M_n are factors of type I_∞, II_∞, III, $n = 1, 2, \ldots, m$, and m is a natural number (some terms could be omitted).*

□

We recall one more important property of the $*$-algebras $LS(M)$.

Proposition 6 ([12]). *Let a von Neumann algebra M be a C^*-product of von Neumann algebras M_i, $i \in \mathrm{I}$, where I is a family of indices, that is, $M = \{\{T_i\}_{i\in\mathrm{I}},\ T_i \in M_i,\ i \in \mathrm{I},\ \sup_{i\in\mathrm{I}} \|T_i\|_{M_i} < \infty\}$ with the coordinate-wise algebraic operations and involution and the $C^\star$-norm, $\|\{T_i\}_{i\in\mathrm{I}}\|_M = \sup_{i\in\mathrm{I}} \|T_i\|_{M_i}$. Then the $*$-algebra $LS(M)$ is $*$-isomorphic to the $*$-algebra $\prod_{i\in\mathrm{I}} LS(M_i)$ (the algebraic operations and the involution in $\prod_{i\in\mathrm{I}} LS(M_i)$ are coordinate-wise.)* □

Let us remark that there is no an analogue of Proposition 1.6 for the algebras $S(M)$. Indeed, let M_n be type III factors, $n = 1, 2, \ldots$, and M be their C^*-product. Then $S(M) = M$ and $LS(M_n) = S(M_n) = M_n$ for all $n = 1, 2, \ldots$. Moreover, by Corollary 1.2, $LS(M) \neq S(M) = M$. Hence, in virtue of Proposition 1.6,

$$\prod_{n=1}^{\infty} S(M_n) = \prod_{n=1}^{\infty} LS(M_n) = LS(M) \neq S(M).$$

The following proposition gives necessary and sufficient conditions for the $*$-algebras $LS(M)$ and M to coincide.

Proposition 7 ([11]). *The following statements are equivalent.*

(i) $LS(M) = M$.

(ii) *M can be represented as a direct sum, $M = \sum_{n=1}^{m} M_n$, where M_n are type I or type III-factors, $n = 1, 2, \ldots, m$, and m is an integer (some terms could be absent).*

□

2 Convergences almost everywhere and locally almost everywhere in the $*$-algebra $LS(M)$.

Let M be an arbitrary von Neumann algebra, $P_f(M)$ a sublattice in $P(M)$ of all finite projections in M.

Definition 1 ([1]). A sequence $\{T_n\}_{n=1}^{\infty} \subset LS(M)$ *converges almost everywhere* to $T \in LS(M)$, denoted by $T_n \xrightarrow{\text{a.e.}} T$, if for any $\varepsilon > 0$ there exists a subsequence $\{E_n\}_{n=1}^{\infty} \subset P(M)$ such that $E_n \uparrow I$, $E_n^{\perp} \in P_f(M)$, $(T_n - T)E_n \in M$ and $\|(T_n - T)E_n\|_M < \varepsilon$ for all $n = 1, 2, \ldots$.

Let M be a commutative von Neumann algebra. Then, as known [13, Part 1, Chapter 7], there exists a measurable space $(\Omega,\, \Sigma,\, \mu)$ with a finite locally complete measure μ such that M is $*$-isomorphic to the $*$-algebra $L_\infty(\Omega,\, \Sigma,\, \mu)$. In this case, the algebra $LS(M) = S(M)$ is $*$-isomorphic to the $*$-algebra $S(\Omega,\, \Sigma,\, \mu)$ of all measurable complex-valued functions defined on $(\Omega,\, \Sigma,\, \mu)$ (the functions that are equal almost everywhere are considered as identical) [1]. The introduced convergence almost everywhere coincides with the convergence almost everywhere with respect to the measure μ in the sense of the measure theory.

It is clear that if $T_n, T \in M$ and $\|T_n - T\|_M \longrightarrow 0$, then $T_n \xrightarrow{\text{a.e.}} T$. The following proposition gives a sufficient condition so that the converse statement holds.

Proposition 8. *Let a von Neumann algebra M be given as a direct sum, $M = \sum_{i=0}^{m} M_i$, where M_0 is a von Neumann algebra of type III, M_i are type I factors, $i = 1, ..., m$, and m is a natural (some terms could be absent). If $T_n, T \in LS(M)$ and $T_n \xrightarrow{a.e.} T$, then $(T_n - T) \in M$ starting with some index, and $\|T_n - T\|_M \longrightarrow 0$ for $n \to \infty$.*

Proof. Any finite projection E in $P(M)$ has the form $E = \sum_{j=1}^{k} P_j$, where P_j are atoms in $P(M)$, $j = 1, 2, \ldots, k$, that is, the reduced von Neumann algebras $P_j M P_j$ are one-dimensional. So, if $Q_n \in P_f(M)$ and $Q_n \downarrow 0$, then $Q_n = 0$ starting with some index n_0. This, together with the definition of convergence almost everywhere, imply that $(T_n - T) \in M$ for $n \geq n_0$ and $\|T_n - T\|_M \longrightarrow 0$ as $n \to \infty$. □

Consider an arbitrary von Neumann algebra of type III, M, such that its center $Z(M)$ does not have atoms. Then the $*$-algebra $LS(Z(M)) = S(Z(M))$ is $*$-isomorphic to the $*$-algebra $S(\Omega,\, \Sigma,\, \mu)$ for a corresponding measurable space with a locally finite continuous measure μ. If $T_n, T \in$

$LS(Z(M)) \subset LS(M)$, $T_n \xrightarrow{\text{a.e.}} T$ in $LS(M)$, then by Proposition 2.1, $(T_n - T) \in Z(M)$ starting with some index, and $\|T_n - T\|_{Z(M)} = \|T_n - T\|_M \longrightarrow 0$ as $n \to \infty$.

Since the measure μ is continuous, there exist $T_n, T \in S(Z(M))$ such that $T_n \to T$ almost everywhere with respect to μ, but $(T_n - T)$ does not belong to M for all $n = 1, 2, \ldots$. This means that convergence of T_n almost everywhere to T in $LS(Z(M))$ does not imply in general the convergence almost everywhere in $LS(M)$.

In this connection, it is natural to modify the notion of convergence almost everywhere in $LS(M)$ so that this convergence would induce the convergence almost everywhere in $LS(Z(M))$.

Definition 2 ([7]). We will call a sequence $\{T_n\}_{n=1}^{\infty}$ in $LS(M)$ convergent *locally almost everywhere* to $T \in LS(M))$, denoted by $T_n \xrightarrow{\text{l.a.e.}} T$, if for any $\varepsilon > 0$ there exist sequences $\{E_n\}_{n=1}^{\infty} \subset P(M)$ and $\{Z_n\}_{n=1}^{\infty} \subset P(Z(M))$ such that $E_n \uparrow I$, $Z_n \uparrow I$, $Z_n E_n^{\perp} \in P_f(M)$, $(T_n - T)E_n \in M$ and $\|(T_n - T)E_n\|_M < \varepsilon$ for all $n = 1, 2, \ldots$.

It is clear that the convergence $T_n \xrightarrow{\text{a.e.}} T$ implies the convergence $T_n \xrightarrow{\text{l.a.e.}} T$ (it is sufficient to take $Z_n = I$, $n = 1, 2, \ldots$). Moreover, it is clear that if M is a factor or a finite von Neumann algebra, convergences almost everywhere and locally almost everywhere coincide. The following theorem gives a relation between convergences almost everywhere and locally almost everywhere for an arbitrary von Neumann algebra M.

Theorem 1. *Let M be an arbitrary von Neumann algebra, $\{T_n\}_{n=1}^{\infty}$, T in $LS(M)$. The following conditions are equivalent:*

(i) $T_n \xrightarrow{l.a.e.} T$.

(ii) There exists a sequence of pairwise orthogonal central projections $\{P_m\}_{m=1}^{\infty}$ such that $\sum_{m=1}^{\infty} P_m = I$ and $T_n P_m \xrightarrow{a.e.} T P_m$, as $n \to \infty$, for each fixed $m = 1, 2, \ldots$.

Proof. (i) $\Longrightarrow$ (ii). Let $T_n \xrightarrow{\text{l.a.e.}} T$, $\varepsilon > 0$ and the projections $\{E_n\}_{n=1}^{\infty} \subset P(M)$, and $\{Z_n\}_{n=1}^{\infty} \subset P(Z(M))$ be such that $E_n \uparrow I$, $Z_n \uparrow I$, $Z_n E_n^{\perp} \in P_f(M)$, $(T_n - T)E_n \in M$ and $\|(T_n - T)E_n\|_M < \varepsilon$ for all $n = 1, 2, \ldots$.

Let $P_1 = Z_1$, $P_m = Z_m - Z_{m-1}$ for $m \geq 2$.

It is clear that $\{P_m\}_{m=1}^{\infty} \subset P(Z(M))$, $\sum_{m=1}^{\infty} P_m = \sup_{m \geqslant 1} Z_m = I$.

Fix m and set $Q_{nm} = E_n P_m + P_m^{\perp}$ for $n \geqslant m$ and $Q_{nm} = 0$ if $n < m$. Then $Q_{nm} \uparrow I$ for $n \to \infty$ and

$$Q_{nm}^{\perp} = I - (E_n P_m + P_m^{\perp}) = P_m - E_n P_m = E_n^{\perp} P_m = (E_n^{\perp} Z_n) P_m \in P_f(M)$$

for $n \geqslant m$. Moreover,

$$(T_nP_m - TP_m)Q_{nm} = (T_n - T)P_mQ_{nm} = (T_n - T)E_nP_m \in M$$

and $\|(T_nP_m - TP_m)Q_{nm}\| < \varepsilon$. This means that $T_nP_m \xrightarrow{\text{a.e.}} TP_m$, as $n \to \infty$, for each fixed $m = 1, 2, \ldots$.

$(ii) \implies (i)$. Let $\{P_m\}_{m=1}^{\infty} \subset P(Z(M))$, $P_mP_n = 0$ for $m \neq n$, $\sum_{m=1}^{\infty} P_m = I$ and $T_nP_m \xrightarrow{\text{a.e.}} TP_m$ as $n \to \infty$ for each fixed $m = 1, 2, \ldots$. Then, for each $\varepsilon > 0$ there is a sequence $\{E_{nm}\}_{n=1}^{\infty} \subset P(M)$ such that $E_{nm} \uparrow I$ for $n \to \infty$, $E_{nm}^{\perp} \in P_f(M)$, $(T_n - T)P_mE_{nm} \in M$ and $\|(T_n - T)P_mE_{nm}\|_M < \varepsilon$ for all $n, m = 1, 2, \ldots$.

Set $Z_n = \sum_{m=1}^{n} P_m$ and $Q_n = \sum_{m=1}^{n} E_{nm}P_m$.

Then $\{Z_n\}_{n=1}^{\infty} \subset P(Z(M))$, $\{Q_n\}_{n=1}^{\infty} \subset P(M)$, $Z_n \uparrow I$, $Q_n \uparrow I$, $Q_n^{\perp}Z_n = \sum_{m=1}^{n} E_{nm}^{\perp}P_m \in P_f(M)$, $(T_n - T)Q_n = \sum_{m=1}^{n}(T_n - T)E_{nm}P_m \in M$ and, since the central supports of operators $(T_n - T)E_{nm}P_m$ are pairwise orthogonal for fixed n, we have

$$\|(T_n - T)Q_n\|_M = \max_{1 \leqslant m \leqslant n} \|(T_n - T)E_{nm}P_m\|_M < \varepsilon.$$

Consequently, $T_n \xrightarrow{\text{l.a.e.}} T$. □

Let us find a class of von Neumann algebras for which the convergences almost everywhere and locally almost everywhere coincide.

Theorem 2. *The following conditions are equivalent.*

(i) Every sequence in $LS(M)$, which is convergent locally almost everywhere, is convergent in $LS(M)$ almost everywhere.

(ii) The von Neumann algebra M can be represented as a direct sum, $M = \sum_{i=0}^{m} M_i$, where M_0 is a finite von Neumann algebra, and M_i are factors of type I_∞, II_∞, or III, $i = 1, 2, \ldots, m$, and m is a natural number (some terms could be missed).

Proof. $(i) \implies (ii)$. Assume that M is not a finite von Neumann algebra and choose a central projection $Q \in Z(M)$ such that $M_0 = Q^{\perp}M$ is a finite von Neumann algebra and QM is a properly infinite von Neumann algebra (it can happen that $Q = I$). Let us show that the center $Z(QM) = QZ(M)$ is a finite dimensional von Neumann algebra.

If this is not the case, then there exists a subsequence $\{Z_n\}_{n=1}^{\infty} \subset P(QZ(M))$ such that $Z_n \uparrow Q$ and $Z_n \neq Q$ for all $n = 1, 2, \ldots$. Then it is clear that $Z_n \xrightarrow{\text{l.a.e.}} Q$ in $LS(M)$ and by (i) we have that $Z_n \xrightarrow{\text{a.e.}} Q$

in $LS(M)$. Hence, there exists a sequence $\{E_n\}_{n=1}^{\infty} \subset P(M)$ such that $E_n \uparrow I$, $E_n^{\perp} \in P_f(M)$, $(Z_n - Q)E_n \in M$ and $\|(Z_n - Q)E_n\|_M < \varepsilon = \frac{1}{2}$ for all $n = 1, 2, \ldots$.

Since $Q - Z_n = Z_n^{\perp}Q$, we see that $(Q - Z_n)E_n = Z_n^{\perp}QE_n$ is a projection such that $\|Z_n^{\perp}QE_n\| < \frac{1}{2}$. Consequently, $Z_n^{\perp}QE_n = 0$ and, hence, $Z_n^{\perp}Q \leqslant E_n^{\perp}$. This means that $Q - Z_n = Z_n^{\perp}Q$ is a nonzero finite projection in QM, which contradicts that the von Neumann algebra QM is properly infinite.

Consequently, the algebra $QZ(M)$ is finite dimensional, that is there exist atoms $Q_1, Q_2, \ldots, Q_m$ in $P(QZ(M))$ such that $\sum_{i=1}^{m} Q_i = I$ and $M_i = Q_iM$ are not finite factors, i.e., they are factors of types I_∞, II_∞, or III. Hence, M is a direct sum, $\sum_{i=0}^{m} M_i$, where $M_0 = Q^{\perp}M$ is a finite von Neumann algebra, and $M_i = Q_iM$ are factors of the above types.

$(ii) \Rightarrow (i)$. Assume that the von Neumann algebra M can be represented as the direct sum $M = \sum_{i=0}^{m} M_i$, where M_0, M_i, $i = 1, 2, \ldots$, are the same as in (ii). By Proposition 1.6,

$$LS(M) = LS(M_0) \bigoplus \sum_{i=1}^{m} LS(M_i).$$

Denote by Q_i the identity element in the von Neumann algebra M_i, $i = 0, 1, 2, \ldots, m$. Let $T_n, T \in LS(M)$ and $T_n \xrightarrow{\text{l.a.e.}} T$ in $LS(M)$ as $n \to \infty$. Then $T_nQ_i \xrightarrow{\text{l.a.e.}} TQ_i$ in $LS(M_i)$ as $n \to \infty$ for any fixed $i = 0, 1, 2, \ldots, m$.

Since M_0 is a finite von Neumann algebra and M_i are factors, we have that $T_nQ_i \xrightarrow{\text{a.e.}} TQ_i$ in $LS(M_i)$ as $n \to \infty$. Since $Q_i \in P(Z(M))$, we see that $T_nQ_i \xrightarrow{\text{a.e.}} TQ_i$ in $LS(M)$ as $n \to \infty$. By Theorem 2.1, $T_n \xrightarrow{\text{a.e.}} T$ in $LS(M)$. □

Remark 1. *Let a von Neumann algebra M be represented as a C^*-product, $M = \prod_{i=1}^{\infty} M_i$, where M_i are factors of types I_∞, II_∞, or III, $i = 1, 2, \ldots$. Then, by Theorem 2.2, the convergences locally almost everywhere and almost everywhere do not coincide in $LS(M)$. In particular, there are von Neumann algebras of countable type for which these convergences do not coincide (recall that a von Neumann algebra M is of a countable type if any family of nonzero pairwise orthogonal projections in $P(M)$ is at most countable).*

Remark 2. *Let M be a factor of type I or III (in this case $LS(M) = M$), $\{T_n\}_{n=1}^{\infty}$, T in M and $T_n \xrightarrow{l.a.e.} T$. Then, for each $\varepsilon > 0$ there exists a sequence $\{E_n\}_{n=1}^{\infty} \subset P(M)$ such that $E_n \uparrow I$, $E_n^{\perp} \in P_f(M)$ (that is, $E_n^{\perp} = 0$ starting with some index n_0), $(T_n - T)E_n \in M$ and $\|(T_n - T)E_n\|_M < \varepsilon$*

(i.e., $\|T_n - T\| < \varepsilon$ as $n \geqslant n_0$). This means that convergence locally almost everywhere coincides with the uniform convergence.

Proposition 9. *Let T_n, $T \in S(Z(M))$. The following conditions are equivalent.*

(i) $T_n \xrightarrow{l.a.e.} T$ *in* $LS(M)$.

(ii) $T_n \longrightarrow T$ *almost everywhere in* $S(\Omega, \Sigma, \mu)$ *(the $*$-algebra $S(Z(M))$ is identified with the $*$-algebra $S(\Omega, \Sigma, \mu)$ and the center $Z(M)$ with the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$).*

Proof. *(i)* $\Rightarrow$ *(ii)*. Without loss of generality, we can assume that $T = 0$. Since $T_n \xrightarrow{\text{l.a.e.}} 0$ in $LS(M)$, by Theorem 2.1 there exists a sequence $\{P_m\}_{m=1}^{\infty}$ of pairwise orthogonal projections such that $\sum_{m=1}^{\infty} P_m = I$ and $T_n P_m \xrightarrow{\text{l.a.e.}} 0$ in $LS(M)$ as $n \to \infty$ for each fixed $m = 1, 2, \ldots$.

Let us fix m and show that $T_n P_m \longrightarrow 0$ almost everywhere in $S(\Omega, \Sigma, \mu)$ as $n \to \infty$.

Choose an arbitrary $\varepsilon > 0$ and choose a sequence $\{E_n\}_{n=1}^{\infty} \subset P(M)$ such that $E_n \uparrow I$, $E_n^{\perp} \in P_f(M)$, and $\|T_n P_m E_n\|_M < \varepsilon$ for all $n = 1, 2, \ldots$.

Denote by $\{E_\lambda(|T_n P_m|)\}$ the spectral family of projections for the operator $|T_n P_m|$. By Lemma 1.1, we have that $E_\varepsilon^{\perp}(|T_n P_m|) \precsim E_n^{\perp}$. Since $E_\varepsilon(|T_n P_m|)$ is a central projection, $E_\varepsilon^{\perp}(|T_n P_m|) \leq E_n^{\perp}$ and, hence, $E_n \leq E_\varepsilon(|T_n P_m|)$ for all $n = 1, 2, \ldots$.

Because $E_n \uparrow I$, we have that $\sup_{n \geq 1}\{inf_{k \geq n} E_\varepsilon(|T_k P_m|)\} = I$ for each $\varepsilon > 0$, that is,

$$\bigcup_{n=1}^{\infty} \Big(\bigcap_{k=n}^{\infty} \{\omega \in \Omega : |T_k(\omega) P_m(\omega)| < \varepsilon\}\Big) = \Omega$$

μ-almost everywhere. This means that $T_n P_m \longrightarrow 0$ almost everywhere in $S(\Omega, \Sigma, \mu)$ as $n \to \infty$ for each fixed $m = 1, 2, \ldots$. Since $\sum_{m=1}^{\infty} P_m = I$, we see that $T_n \longrightarrow 0$ almost everywhere in $S(\Omega, \Sigma, \mu)$.

(ii) $\Rightarrow$ *(i)*. Let $T_n \longrightarrow 0$ almost everywhere in $S(\Omega, \Sigma, \mu)$. Then, for every $\varepsilon > 0$, the following holds:

$$\bigcup_{n=1}^{\infty} \Big(\bigcap_{k=n}^{\infty} \{\omega \in \Omega : |T_k(\omega)| < \varepsilon\}\Big) = \Omega$$

μ-almost everywhere.

Denote by Z_n the central projection in $Z(M)$ corresponding to the set $(\bigcap_{k=n}^{\infty} \{\omega \in \Omega : |T_k(\omega)| < \varepsilon\}) \in \Sigma$. It is clear that $Z_n \uparrow I$ and, for $E_n = Z_n$, we have that $Z_n E_n^{\perp} = 0 \in P_f(M)$, $\|T_n E_n\|_M = \|T_n Z_n\|_{Z(M)} < \varepsilon$ for all $n = 1, 2, \ldots$. This means that $T_n \xrightarrow{\text{l.a.e.}} 0$ in $LS(M)$. $\square$

Let M be an arbitrary commutative von Neumann algebra. Then as was noted above, there exists a measurable space (Ω, Σ, μ) with a locally finite complete measure μ such that M is $*$-isomorphic to the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$ and the $*$-algebra $LS(M) = S(M)$ is $*$-isomorphic to the $*$-algebra $S(\Omega, \Sigma, \mu)$. So, together with a well-known convergence in $S(\Omega, \Sigma, \mu)$ with respect to measure, we also consider the convergence locally with respect to measure. This convergence is defined as follows: a sequence $\{f_n\}_{n=1}^{\infty} \subset S(\Omega, \Sigma, \mu)$ *converges locally with respect to measure* to $f \in S(\Omega, \Sigma, \mu)$ as $n \to \infty$ if $f_n \chi_A \longrightarrow f \chi_A$ with respect to measure for any set $A \in \Sigma$ with $\mu(A) < \infty$, where χ_A is a characteristic function of the set A.

A similar convergence can be also defined in the algebra $LS(M)$ in the case of an arbitrary von Neumann algebra M.

Denote by φ a $*$-isomorphism of the center $Z(M)$ of the von Neumann algebra M to the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$ and by $S_\infty^+(\Omega, \Sigma, \mu)$ the set of all measurable functions $f : \Omega \longrightarrow [0, \infty]$ (functions that are equal almost everywhere are identified). It was shown in [1] that there exists a mapping

$$d : P(M) \longrightarrow S_\infty^+(\Omega, \Sigma, \mu)$$

such that

(i) $d(P) = 0$ if and only if $P = 0$;

(ii) $d(P)$ is finite almost everywhere if and only if the projection P is finite;

(iii) $d(P + Q) = d(P) + d(Q)$ if $PQ = 0$;

(iv) $d(U^*U) = d(UU^*)$ for any partial isometry $U \in M$;

(v) $d(ZP) = \varphi(Z) d(P)$ for all $Z \in P(Z(M))$ and $P \in P(M)$;

(vi) if P_α, $P \in P(M)$ and $P_\alpha \uparrow P$, then $d(P) = \sup_\alpha d(P_\alpha)$.

A mapping $d : P(M) \longrightarrow S_\infty^+(\Omega, \Sigma, \mu)$ satisfying the properties (i)—(vi) is called a dimension function on $P(M)$.

For each $\varepsilon > 0$ and $A \in \Sigma$ satisfying $\mu(A) < \infty$, we set

$$V(A, \varepsilon) = \{T \in LS(M) : \text{ there exists } P \in P(M) \text{ such that } TP \in M, \\ \|TP\|_M < \varepsilon, \text{ and } \mu(A \cap \{\omega \in \Omega : d(P^{\perp})(\omega) > \varepsilon\}) < \varepsilon\}.$$

Theorem 3 ([7]). *(i) The system of the sets*

$$\{\{T + V(A, \varepsilon)\} : T \in LS(M), \ \varepsilon > 0, \ A \in \Sigma, \ \mu(A) < \infty\} \qquad (1)$$

defines in $LS(M)$ a Hausdorff vector topology t for which sets (14.1) *form a base of neighborhoods of the operator $T \in LS(M)$.*

(ii) $(LS(M), t)$ is a complete uniform space with respect to the dimension induced by the topology t.

(iii) The involution is continuous, and the multiplication in $(LS(M), t)$ is continuous in the totality of the variables (that is $(LS(M), t)$ is a topological $*$-algebra).

(iv) The topology t is metrizable if and only if the Boolean algebra $P(Z(M))$ is of countable type, that is, any family of nonzero pairwise orthogonal projections in $P(Z(M))$ is at most countable.

(v) If $\{T_\alpha\}_{\alpha \in J}, T \subset LS(M)$, then the net T_α converges to T in the topology t (denoted by $T_\alpha \xrightarrow{t} T$) if and only if $E_\lambda^\perp(|T_\alpha - T|) \xrightarrow{t} 0$ for any $\lambda > 0$, where $\{E_\lambda(|T_\alpha - T|)\}$ is a spectral family of projections for $|T_\alpha - T|$. In particular, $T_\alpha \xrightarrow{t} T$ if and only if $|T_\alpha - T| \xrightarrow{t} 0$.

(vi) If $\{P_n\}_{n=1}^\infty \subset P(M)$, then $P_n \xrightarrow{t} 0$ if and only if $\chi_A d(P_n) \longrightarrow 0$ with respect to the measure μ for each $A \in \Sigma$ with $\mu(A) < \infty$.

□

It was found in [7] that the topology t does not change if the measure μ is replaced with an equivalent measure and the dimension function d with another dimension function.

Convergence in the topology t is called a *convergence locally in measure.*

It follows from the definition of the topology t that the convergence of a net $\{T_\alpha\}_{\alpha \in J}$ to T locally in measure means that for any $\varepsilon > 0$ and $A \in \Sigma, \ \mu(A) < \infty$, there exists $\alpha_0 = \alpha(\varepsilon, A)$ such that, for each $\alpha \geq \alpha_0$, there exists a projection $P(\alpha) \in P(M)$ satisfying

$$\|(T_\alpha - T)P(\alpha)\|_M < \varepsilon \qquad (2)$$

and

$$\mu(A \cap \{\omega \in \Omega : \ d(I - P(\alpha))(\omega) > \varepsilon\}) < \varepsilon. \qquad (3)$$

If inequality (14.2) is replaced with the inequality

$$\|P(\alpha)(T_\alpha - T)P(\alpha)\|_M < \varepsilon, \qquad (14.2')$$

then it is said that the net $\{T_\alpha\}_{\alpha\in J}$ *converges to* T *two-side locally in measure.*

It is easy to see that the two-side convergence in measure is equivalent to the convergence in the vector topology in $LS(M)$, with the base of neighborhoods of zero formed by the sets

$$\begin{aligned} W(A,\,\varepsilon) = \{T \in LS(M):\ &\text{there exists } P \in P(M) \\ &\text{such that } PTP \in M,\ \|PTP\|_M < \varepsilon \\ &\text{and } \mu(A \cap \{\omega \in \Omega:\ d(P^\perp)(\omega) > \varepsilon\}) < \varepsilon\}, \\ &\text{where } \varepsilon > 0,\ A \in \Sigma,\ \mu(A) < \infty. \end{aligned}$$

In fact, this vector topology coincides with the topology t, which is directly implied by the following proposition.

Proposition 10 ([11]).

$$V(A,\varepsilon) \subset W(A,\varepsilon) \subset V(A, 2\varepsilon)$$

for any $\varepsilon > 0$, $A \in \Sigma$, $\mu(A) < \infty$.

If there exists a faithful normal semi-final trace τ on a von Neumann algebra M, then, for the $*$-algebra $LS(M)$, one can consider convergence in measure induced by the trace τ, see, e.g. [2, 3]. This convergence coincides with the convergence in the vector topology t_τ in $LS(M)$, with a base of neighborhoods of zero formed by the sets

$$\begin{aligned} V(\varepsilon,\,\delta) = \{T \in LS(M):\ &\text{there exists } P \in P(M) \\ &\text{such that } TP \in M,\ \|TP\|_M < \varepsilon, \tau(P^\perp) < \delta\}, \end{aligned}$$

where $\varepsilon,\,\delta > 0$.

Proposition 11 ([11]). *Let* τ *be a faithful normal semi-finite trace on a von Neumann algebra* M*. Then we have the following.*

(i) If $\{E_n\}_{n=1}^\infty \subset P(M)$ *and* $\tau(E_n) \longrightarrow 0$*, then* $E_n \xrightarrow{t} 0$*. Conversely, if* $E_n \xrightarrow{t} 0$ *and* $\tau(I) < \infty$*, then* $\tau(E_n) \longrightarrow 0$*.*

(ii) If $\{T_n\}_{n=1}^\infty, T \subset LS(M)$ *and* $T_n \xrightarrow{t_\tau} T$*, then* $T_n \xrightarrow{t} T$*.*

(iii) If $\tau(I) < \infty$*, then the topologies* t *and* t_τ *coincide.*

Remark 3. *If the trace τ is not finite, then the convergence $T_n \xrightarrow{t} T$, in general, does not imply the convergence $T_n \xrightarrow{t_\tau} T$ even for commutative von Neumann algebras.*

Example 1. *Consider the von Neumann algebra*

$$M = l_\infty = \{\{c_n\}_{n=1}^\infty : \ c_n \in \boldsymbol{C}, n = 1,\, 2, \ldots, \sup_{n \geq 1} |c_n| < \infty\}.$$

Set $\tau(\{c_n\}) = \sum_{n=1}^\infty c_n$ and $\tau_1(\{c_n\}) = \sum_{n=1}^\infty 2^{-n} c_n$, where $\{c_n\} \in l_\infty, c_n \geq 0$.

Then τ is a faithful normal trace on M that is semi-finite but not finite, and τ_1 is a faithful normal finite trace on M.

Consider a sequence of projections, $E_n = (\underbrace{0, 0, \ldots, 0}_{n}, 1, 1, \ldots)$, in l_∞, decreasing to zero. Then $\tau_1(E_n) = \sum_{k=n+1}^\infty 2^{-k} = 2^{-n} \to 0$ as $n \to \infty$ and, by Proposition 2.4(i), $E_n \xrightarrow{t} 0$.

However, $\tau(\{E_n > \frac{1}{2}\}) = +\infty$ for all $n = 1, 2, \ldots$, and so $E_n \stackrel{t_\tau}{\nrightarrow} 0$. □

Remark 4. *Let M be a factor. Then $Z(M) = \boldsymbol{C} = L_\infty(\{\omega\},\, \Sigma,\, \mu)$, where $\Sigma = \{\emptyset,\, \{\omega\}\}$, $\mu(\{\omega\}) = 1$. In this case, the dimension function d is a faithful normal semi-finite (finite) trace on M if M is of type I_∞, II_∞ (correspondingly, I_n, II_1), and $d(E) = +\infty$ for all nonzero $E \in P(M)$ if M is of type III.*

So, if $\varepsilon \in (0, 1)$, $A = \{\omega\}$, we have that

$$V(A,\, \varepsilon) = \{T \in LS(M) : \text{ there exists } P \in P(M) \text{ such that } TP \in M, \\ \|TP\|_M < \varepsilon, \text{ and } d(P^\perp) < \varepsilon\}.$$

In other words, if M is of type III, then

$$V(A,\, \varepsilon) = \{T \in M : \|T\|_M < \varepsilon\},$$

that is the convergence locally in measure coincides with uniform convergence, and if M is of type I or II, then convergence locally in measure coincides with convergence in measure induced by the trace d.

Remark 5. *If $M = \mathcal{B}(H)$ is a factor of type I, then convergence locally in measure coincides with uniform convergence.*

Indeed, let $\tau = tr$ be the canonical trace on $\mathcal{B}(H)$, $T_n, T \in \mathcal{B}(H)$, and $T_n \xrightarrow{t} T$ (note that, by Proposition 1.7, $LS(M) = S(M) = M = \mathcal{B}(H)$).

By Theorem 2.3 (v), (vi), we have that $tr(E_\lambda^\perp(|T_n - T|)) \longrightarrow 0$ *as* $n \to \infty$ *for any* $\lambda > 0$. *Consequently,* $E_\lambda^\perp(|T_n - T|) = 0$ *starting with some index* $n(\lambda)$. *This means that* $\|T_n - T\|_M = \||T_n - T|\|_M \leq \lambda$ *for* $n \geq n(\lambda)$, *that is,* $\|T_n - T\|_M \longrightarrow 0$ *as* $n \to \infty$.

Remark 6. *If* $T_n, T \in S(Z(M))$, *then* $T_n \xrightarrow{t} T$ *if and only if* $T_n \longrightarrow T$ *in the measure* μ *for each* $A \in \Sigma$ *with* $\mu(A) < \infty$ *(we identify* $S(Z(M))$ *with the* $*$*-algebra* $S(\Omega, \Sigma\ \mu)$*).*

Indeed, if $\{E_\lambda(|T_n - T|)\}$ *is a spectral family of projections for the operator* $|T_n - T|$, *then by Theorem 2.3 (v),* $T_n \xrightarrow{t} T$ *if and only if* $E_\lambda^\perp(|T_n - T|) \xrightarrow{t} 0$ *for any* $\lambda > 0$. *Since* $T_n, T \in S(Z(M))$, *we have that* $E_\lambda(|T_n - T|) \in P(Z(M))$ *for all* $\lambda > 0$. *By Theorem 2.3 (vi),* $E_\lambda^\perp(|T_n - T|) \xrightarrow{t} 0$ *if and only if* $\chi_A E_\lambda^\perp(|T_n - T|)d(I) = \chi_A d(E_\lambda^\perp(|T_n - T|)) \longrightarrow 0$ *in the measure* μ *for each* $A \in \Sigma$ *with* $\mu(A) < \infty$, *where we identify* $Z(M)$ *with the* $*$*-algebra* $L_\infty(\Omega, \Sigma, \mu)$ *(see the definition of the dimension function* d*).*

Consequently, $T_n \xrightarrow{t} T$ *if and only if* $E_\lambda^\perp(|T_n - T|)$ *converges to zero in the measure* μ *for each* $A \in \Sigma$ *with* $\mu(A) < \infty$ *for all* $\lambda > 0$. *The latter condition, clearly, is equivalent to the convergence* $T_n \longrightarrow T$ *in the measure* μ *for each* $A \in \Sigma$ *with* $\mu(A) < \infty$.

The following theorem gives a criterion for the convergences locally almost everywhere and locally in measure to coincide in $LS(M)$.

Theorem 4. *The following conditions are equivalent.*

(i) For $\{T_n\}_{n=1}^\infty$ *and* T *in* $LS(M)$, $T_n \xrightarrow{l.a.e.} T$ *if and only if* $T_n \xrightarrow{t} T$.

(ii) The von Neumann algebra M *can be represented as a* C^**-product,* $M = \prod_{i \in J} M_i$, *where* M_i *are factors of types I or III,* $i \in J$, *and* J *is an index set.*

Proof. $(i) \Longrightarrow (ii)$. Identify the center $Z(M)$ of the von Neumann algebra M with the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$, and the $*$-algebra $LS(M)$ with the $*$-algebra $S(\Omega, \Sigma, \mu)$. If the space with the measure (Ω, Σ, μ) is not atomic, then there exists a set $A \in \Sigma$ with $0 \neq \mu(A) < \infty$ such that the Boolean algebra $Q(A)Z(M)$ does not have atoms, where $Q(A)$ is the central projection in $Z(M)$ corresponding to the set A. In this case, as shown in [14], there exists a sequence $\{G_n\} \subset P(Q(A)Z(M))$ such that $\mu(G_n) \longrightarrow 0$, but $\{G_n\}$ does not converge to zero μ-almost everywhere. It follows from Remark 6 that $G_n \xrightarrow{t} 0$. So, by assumption (i), we have $G_n \xrightarrow{\text{l.a.e.}} 0$. But then, by Proposition 2.2, $G_n \longrightarrow 0$ almost everywhere in $S(\Omega, \Sigma, \mu)$, which is not true. Consequently, the Boolean algebra $P(Z(M))$ of central projections in M is atomic.

Let $\{Q_i\}_{i\in J}$ be the set of all atoms in $P(Z(M))$ and $M_i = Q_iM$, $i \in J$. Then M_i is a factor for each $i \in J$ and the von Neumann algebra M is $*$-isomorphic to the C^*-product $\prod_{i\in J} M_i$ (this $*$-isomorphism is given by the mapping $\psi : M \longmapsto \prod_{i\in J} M_i$, where $\psi(T) = \{Q_iT\}_{i\in J}$, $T \in M$).

Let τ_i be a faithful normal trace on M_i (if M_i is of type III, then $\tau_i(0) = 0$ and $\tau_i(T) = +\infty$ for any positive operator $T \in M_i$).

The mapping $d : M \longmapsto S^+_\infty(\Omega, \Sigma, \mu)$ defined by the formula $d(E) = \{\tau_i(EQ_i)\}_{i\in J}$ is a dimension function on M (since the Boolean algebra $P(Z(M))$ is atomic, Ω can be identified with J and Σ with the σ-algebra of all subsets of J, with $\mu(i) < \infty$ for all $i \in J$).

Suppose that there exists $i_0 \in J$ such that M_{i_0} has type II_1 or II_∞. Then M_i contains a commutative von Neumann subalgebra B such that (B, τ_{i_0}) is $*$-isomorphic to $(L_\infty([0, 1]), m)$, where m is a linear Lebesgue measure on the line segment $[0, 1]$. Using the proof of Theorem 8 in [14] we see that there exists a sequence $\{E_n\}_{n=1}^\infty \subset P(M_{i_0})$ such that $\tau_{i_0}(E_n) \longrightarrow 0$, but $\{E_n\}_{n=1}^\infty$ does not converge to zero in $S(M_{i_0})$ almost everywhere. It follows from Theorem 2.3 (vi) that $\hat{E}_n \xrightarrow{\text{t}} 0$, where $\hat{E}_n = \{T_i^{(n)}\}_{i\in J} \in P(M)$, $T_i^{(n)} = 0$ for $i \neq i_0$ and $T_{i_0}^{(n)} = E_n$. Consequently, by assumption (i), $\hat{E}_n \xrightarrow{\text{l.a.e.}} 0$, and so, $Q_i\hat{E}_n \xrightarrow{\text{l.a.e.}} 0$, which implies that $E_n \xrightarrow{\text{l.a.e.}} 0$ in $LS(M_{i_0}) = S(M_{i_0})$. Since M_{i_0} is a factor, $E_n \longrightarrow 0$ in $S(M_{i_0})$ almost everywhere, which is not true. This contradiction shows that each factor M_i has type I or III, $i \in J$.

$(ii) \Rightarrow (i)$. Let a von Neumann algebra M be represented as a C^*-product, $M = \prod_{i\in J} M_i$, where M_i are factors of types I or III. To prove the implication $(ii) \Rightarrow (i)$, it is sufficient to show that if $\{T_i^{(n)}\}_{i\in J} = T_n \in LS(M) = \prod_{i\in J} LS(M_i)$ and $T_n \xrightarrow{\text{t}} 0$, then $T_n \xrightarrow{\text{l.a.e.}} 0$.

Set $Q_j = \{E_i\}_{i\in J} \in P(Z(M))$, where $E_i = 0$ for $i \neq j$ and $E_j = I_{M_j}$ is the identity element in the algebra M_j. Assume that $T_n \xrightarrow{\text{t}} 0$. Then, by Theorem 2.3 (iii), $Q_jT_n \xrightarrow{\text{t}} 0$ for each $j \in J$.

Fix $\varepsilon > 0$ and set

$$Z_n = \sup\{Q_i : \ \|T_i^{(k)}\|_{M_i} < \varepsilon \quad \text{for all } k \geq n\}, \qquad n = 1, 2, \ldots.$$

It is clear that $Z_n \in P(Z(M))$ and $Z_n \leq Z_{n+1}$, $n = 1, 2, \ldots$.

Let $Z_0 = \sup_{n\geq 1} Z_n$. If $Z_0 \neq I$, then there is $i_0 \in J$ such that $Z_0Q_{i_0} = 0$.

On the other hand, it follows from the convergence $\|T_i^{(k)}\|_{M_i} \longrightarrow 0$ as $k \to \infty$ that $Q_{i_0} \leq Z_{n(\varepsilon)} \leq Z_0$ for some index $n(\varepsilon)$, which contradicts the identity $Z_0Q_{i_o} = 0$. Consequently, $Z_n \uparrow I$. Set $E_n = Z_n$,

$n = 1, 2, \ldots$. Then $E_n \uparrow I$, $Z_n E_n^{\perp} = 0 \in P_f(M)$, $\|T_n E_n\|_M = \|T_n Z_n\|_m = \sup_{i:\ Q_i \leq Z_n} \|T_i^{(n)}\|_{M_i} \leq \varepsilon$.

This means that $T_n \xrightarrow{\text{l.a.e.}} 0$. □

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