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Best wavelet bases for image compression generated by rational symbols

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Abstract

We present the results of a numerical search for a wavelet basis with the best performance for image compression. Wavelets of multiparametric families of wavelets with rational masks were considered as candidates for the best basis. The results of the modeling are as follows.

1. The best biorthogonal bases coincide with the classical 9/7 bases for the case of even symmetric wavelets and with 22/14 coiflets for the odd case.

2. The optimality of the basis depends only on the image resolution. Adaptation to a certain image practically do not improve the performance.

Keywords: Orthogonal Greedy Algorithm, wavelet frame, image compression.

1 Introduction

The problem of finding the best wavelet bases for image compression has a long history. An enormous number of papers was published on this topic. We mention only a few of them [1–3]. Currently, when JPEG2000 is accepted as the international image compression standard much less attention is paid to this topic. The famous 9/7 and 5/3 bases from [1] were taken as the state-of-the-art bases for the lossy and lossless compression. While there are a few bases which can compete with the mentioned ones (cf, [2]), it is clear that staying in the framework of the standard dyadic separable wavelets the performance cannot be improved significantly. At the same time, a lot of questions are still open.

One of the crucial question is which properties of the bases guarantee the high performance in an image approximation (compression). There are the rules of thumb requiring the smoothness of the wavelet basis for reconstruction (a synthesis operator) agreeing with the smoothness of images and as many as possible vanishing moments for a dual wavelet system (an analysis operator) used for the computation of wavelet coefficients. The mentioned classical wavelets satisfy those rules. However, it is not clear why a lot of other wavelet systems satisfying the same properties have much lower performance. A comprehensive answer to this question probably will be found if the researchers understand what is an image. However, the nature of images is very complicated and an adequate model of images has not be found yet. Thus, the smoothness and the vanishing moments provide us with a necessary (not sufficient) condition.

Another question tightly associated with the first one is about opportunity to improve wavelet bases performance in image compression essentially. Basing on the state-of-the-art knowledge this perspective from the practical point of view looks very cloudy. As it was mentioned above, impossibility of such improvement is due to the failure to find better bases in spite of the extensive search performed by many researchers.

In this note, we are not going to give any comprehensive answers those questions. This is just a report about numerical experiments emphasizing on some special aspects of the problems. We suggest a simple procedure allowing to extend the search range for best bases significantly. The reported numerical results bring a lot of questions rather than give answers. Some of the results implicitly contradict to widely accepted point of views. However, we hope that this report may be a useful brick in construction of image models.

We arrive at two main conclusions. First of all, the best biorthogonal bases have been already found. Secondly, the best basis practically does not depend on an image.

2 Definitions and Methods

We refer the reader to the book [4] for a comprehensive introduction to the theory of wavelet bases. Only definitions essential for this note are given below.

Associated with Multiresolution Analysis (MRA) biorthogonal wavelets can be generated by two formal Laurent series $H_0(z) = \sum_{\mathbb{Z}} h_k z^k$ and $\tilde{H}_0(z) = \sum_{\mathbb{Z}} \tilde{h}_k z^k$. In this note, the coefficient sequences h_k and $\tilde{h}_k$ decay exponentially. $H_0(z)$ and $\tilde{H}_0(z)$ are called *symbols* of mutually dual MRAs (or, to be more precise, dual *scaling functions*). Two derived functions $H_1(z) = z\tilde{H}_0(-1/z)$ and $\tilde{H}_1(z) = zH_0(-1/z)$ are called *wavelet symbols.*

To provide the perfect reconstruction property the condition

$$H_0(z)\tilde{H}_0(1/z) + H_0(-z)\tilde{H}_0(-1/z) = 1. \tag{1}$$

as well as low-pass filter condition $H_0(1) = 1$ and $\tilde{H}_0(1) = 1$ are imposed. In this case, for the matrices

$$M(z) := \begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{pmatrix} \quad \text{and} \quad \tilde{M}(z) := \begin{pmatrix} \tilde{H}_0(z) & \tilde{H}_0(-z) \\ \tilde{H}_1(z) & \tilde{H}_1(-z) \end{pmatrix}$$

the identity

$$M^T(z)\tilde{M}(1/z) = I$$

takes place. In particular, representing the data $\{c_k\}$ in the form of formal Laurent series $x(z) := \sum c_k z^k$, we have

$$\begin{pmatrix} y_0(z^2) \\ y_1(z^2) \end{pmatrix} = \tilde{M}(1/z) \begin{pmatrix} x(z) \\ x(-z) \end{pmatrix}, \quad \begin{pmatrix} x(z) \\ x(-z) \end{pmatrix} = M^T(z) \begin{pmatrix} y_0(z^2) \\ y_1(z^2) \end{pmatrix}, \tag{2}$$

where coefficients of the Laurent series $y_0(z^2)$ and $y_1(z^2)$ are called *low-pass* and *high-pass* coefficients of the decomposition correspondingly.

In Electrical Engineering the set of functions $H_0(z)$, $\tilde{H}_0(z)$, $H_1(z)$, $\tilde{H}_1(z)$ are called *a filter bank* (FB) and the functions constituting the filter bank are called *filters.*

A wavelet transform is organized in multilevel manner when the the low-pass output $y_0(z^2)$ of a filter bank transform is used after downsampling $y_0(z^2) \mapsto x^1(z)$ as an input for the next filter bank transform and so on as many times as necessary.

We note that, in fact, the decomposition—reconstruction formulas consist of convolutions. Therefore, we have less computational costs when a filter bank consists of Laurent polynomials of low degree. For this reason, polynomial filter banks became very popular in industrial applications.

It is well-known that the implementation of rational filter banks of the form

$$H(z) = \frac{P(z)}{Q(z)}$$

can be implemented with, so-called, recursive filters with at least the same order of computational efficiency as polynomial filters. At the same time, rational filters bring enormous amount of the flexibility.

While the first non-trivial wavelet basis constructed by Strömberg ([5]) is associated with a rational filter bank, the first comprehensive study of this opportunity was conducted by C. Herley and M. Vetterli [6]. In particular, they showed that rational filter banks may combine the symmetry, several vanishing moments, and the paraunitarity ($M(z) = \tilde{M}(z)$) of filter banks. A few years later, rational FBs underwent more detail study [7–11]. Their high efficiency in applications was shown, e.g, in [12, 13].

Now we explain what is a source of the higher flexibility of rational FBs against polynomial ones. The reason lies in the necessity to solve the equation (1) for providing the perfect reconstruction property. While there are a lot of polynomial solutions (even with a fixed degree) to (1), equation (1) is very restrictive to provide a combination of the symmetry and an appropriate number of vanishing moments.

In the rational case, we are just starting from two arbitrary Laurent polynomials $P(z)$ and $\tilde{P}(z)$, $P(1) \neq 0$, $\tilde{P}(1) \neq 0$. Let

$$R(z) := P(z)\tilde{P}(1/z) + P(-z)\tilde{P}(-1/z),$$

then obviously $R(z)$ depends only on z^2, i.e., $R(z) = Q_0(z^2)$, and if $P(z)$ and $\tilde{P}(z)$ are both with whole- or half-point symmetry, then $Q_0(z)$ has the whole-point symmetry. In particular, the last property means that a symmetric factorization $Q_0(z) = Q(z)\tilde{Q}(z)$ (maybe not unique) exists. Thus, the filters

$$H_0(z) = \frac{P(z)}{Q(z^2)} \quad \text{and} \quad \tilde{H}_0(z) = \frac{\tilde{P}(z)}{\tilde{Q}(z^2)}$$

are solutions to (1). While not all FBs satisfying (1) generate genuine wavelet bases, the degree of freedom for rational filter banks is not comparable with polynomial constructions.

Let us estimate numerically the number of degrees of freedom for the described FBs when low-pass filters are symmetric. For the whole-point symmetry, convolution kernels generated by $P(z)$ and $\tilde{P}(z)$ have odd lengths $2n+1$ and $2m+1$. Due to the symmetry, we are free to choose only $n+1$ and $m+1$ coefficients. In addition, multiplication of the polynomial by constants do not bring new FBs. So we have only $d = n + m$ degrees of freedom.

Obviously, for half-point symmetric FBs with the polynomials at the numerators generating convolutions of lengths $2n$ and $2m$, we have $d = n + m - 2$ degrees of freedom.

In particular, in our numerical experiments, we optimized (anti)symmetric wavelets for 3 cases:

- 8/4 ($n = 4$, $m = 2$) rational wavelets, $d = 4$;
- 10/6 ($n = 5$, $m = 3$) rational wavelets, $d = 6$;
- 9/7 ($n = 4$, $m = 3$) rational wavelets, $d = 6$.

For example, for the 8/4 FB, we have

$$H_0(z) = \frac{a_0(1+z) + a_1(z^{-1}+z^2)}{(1+qz^2)(1+q^{-1}z^{-2})}, \tag{3}$$

$$\tilde{H}_0(z) = \frac{\tilde{a}_0(1+z) + \tilde{a}_1(z^{-1}+z^2) + \tilde{a}_2(z^{-1}+z^3) + \tilde{a}_3(z^{-2}+z^4)}{(1+\tilde{q}z^2)(1+\tilde{q}^{-1}z^{-2})} \tag{4}$$

3 Modeling settings and results

We used simple scheme of a random search of the best basis among wavelets of 4- or 6-parametric families (see above). The initial point was also chosen randomly.

The search was performed from coarse approximation to follow-up refinements of the range and the step of the search. It should be mentioned that the problem has many local extrema. For this reason, the method does not give the same result for each trial. However, for overwhelming majority of trials the optimal wavelets coincide. In addition, there is some kind of "inverse instability" when relatively large fluctuations of FB coefficients do not result in an essential change of the wavelet bases.

For our experiments we used standard grayscale test images of size 512×512 like *Lena, Barbara, Boat, Goldhill* and so on (totally 12 images). We started from 4- or 5-level decomposition with follow-up uniform quantization (with a usual extra dead zone around 0). Wavelet coefficients underwent entropy encoding with either SPIHT ([14]) or PACC2 ([12]).

Now we describe an optimization goal function. For a fixed length of an entropy encoder output, Euclidean deviation of the approximation from the original image was computed. To be more precise, we computed PSNR value in dBs which is defined by the formula

$$PSNR = 10 \log_{10} \frac{512^2 255^2}{\sum (x_i - \hat{x}_i)^2},$$

where $\{x_i\}$ and $\{\hat{x}_i\}$ are values of the luminosity of the original and approximating images.

While PSNR-criterion is quite far from the "metric" of a human eye, it is traditionally used in image processing.

The length of the code was chosen from the requirement to have 30-40 dB of PSNR.

Maximization of PSNR was conducted for separate images. In addition, we optimized FBs for average PSNR as a goal function taken over all 12 images.

The results of the modeling are as follows.

1. The optimal FB practically does not depend on the image. This fact contradicts the widely accepted point of view that for images with different contents the choice of the basis may bring significant improvement. The actual improvement is within 0.1dB. By analogy with stochastic processes, this property may be called *ergodic*. Of course, we have this property only for natural still images taken with a picture camera. Artificial images (like a cartoon) require different methods.

2. The graphs of the optimal 8/4 wavelets and coefficients of the rational FB (from formulas (3) and (4)) are given on Fig. 1 (the solid line) and in Table 1. It turns out that the optimal bases were already found among polynomial FBs. The polynomial 22/14 implementation of the 8/4 FB was found in [2] (the dotted line on Fig. 1). Those wavelets with 7 and 5 vanishing moments satisfy Coifman property. While in some applications the property of a FB to be polynomial is preferable, the rational 8/4 FB has much less computational complexity.

3. The extension of the search up to 6-parametric family of 10/6 wavelets do not bring any visible improvement of the performance. So we arrive at the conclusion that staying in the described framework of compression we cannot improve the performance due to new wavelet bases.

4. The optimization of 9/7 rational FBs gives wavelet bases practically non-distinguishable from the classical 9/7 wavelets. The performance of the optimal basis is within a few hundredth of dB from the polynomial FB.

$\sqrt{2}H_0(z)$			$\sqrt{2}\tilde{H}_0(z)$				
a_0	a_1	q	$\tilde{a}_0$	$\tilde{a}_1$	$\tilde{a}_2$	$\tilde{a}_3$	$\tilde{q}$
0.690304	0.254762	0.156082	0.736905	0.280818	0.030724	0.043537	0.242698

Table 1: Coefficients for the optimal 8/4 bases generated by rational symbols.

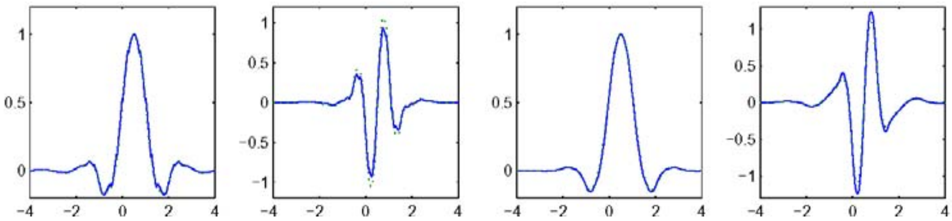

Figure 1: Optimal bi-orthogonal 8/4 and 22/14 wavelet bases for 512×512 images: a) decomposition pair (left); b) reconstruction pair (right). Solid lines correspond to 8/4 wavelets.

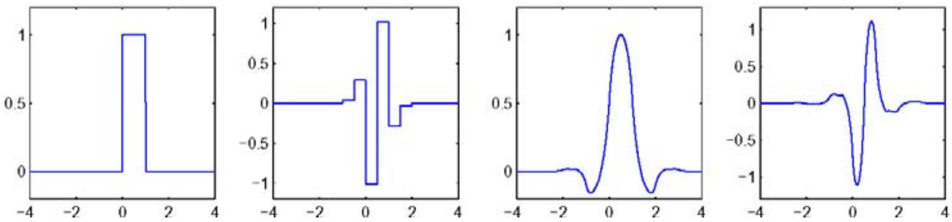

Figure 2: Optimal bi-orthogonal wavelet bases for QCIF images: a) decomposition pair (left); b) reconstruction pair (right).

While the dependency of optimal bases on a particular image is very weak, the dependency on image resolution is much more explicit. In [12], we optimized basis for a video codec working with, so-called, QCIF format (172×144 pixels) intended for very low bit rate compression. The graphs of those optimal wavelet bases are shown on Fig. 2. It is easy to see that they are quite different.

We note that the modeling was also conducted for different goal functions. Among of them we considered Hausdorff and ℓ^1 approximations. In all cases, there was no visible difference between optimal bases.

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