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Ring homeomorphisms and the Beltrami equation

Vladimir Ryazanov[1], Uri Srebro[2] and Eduard Yakubov[3,*]

[1] *Institute of Applied Mathematics and Mechanics, NAS of Ukraine, ul. Roze Luxemburg 74, Donetsk 83114, Ukraine*
[2] *Technion - Israel Institute of Technology, Haifa 32000, Israel*
[3] *Holon Institute of Technology, 52 Golomb St., Holon 58102, Israel*
[*] *Corresponding author: yakubov@hit.ac.il*
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Abstract

We study ring homeomorphisms and, on this base, obtain a series of theorems on existence of the so-called ring solutions for the Beltrami equation. In particular, we give new existence criteria in terms of functions of finite mean oscillation.

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1 Introduction

Let D be a domain in the complex plane $\mathbb{C}$, i.e., open and connected subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. The *Beltrami equation* is the equation of the form

$$f_{\overline{z}} \,=\, \mu(z) \cdot f_z \tag{1.1}$$

where $f_{\overline{z}} = \overline{\partial} f = (f_x + if_y)/2$, $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f in x and y, correspondingly. The function μ is called the *complex coefficient* and

$$K_\mu(z) \,=\, \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

the *maximal dilatation* or in short the *dilatation* of the equation (1.1).

Recall that a mapping $f : D \to \mathbb{C}$ is *absolutely continuous on lines*, abbr. $f \in$ACL, if, for every closed rectangle R in D whose sides are parallel to the coordinate axes, $f|R$ is absolutely continuous on almost all line segments in R which are parallel to the sides of R. In particular, f is ACL if it belongs to the Sobolev class $W^{1,1}_{loc}$, see e.g. [1], p. 8. Note that, if $f \in$ ACL, then f has partial derivatives f_x and f_y a.e. and, thus, by the well-known Gehring-Lehto theorem every ACL homeomorphism $f : D \to \mathbb{C}$ is totally differentiable a.e., see [2] or [3], p. 128. For a sense-preserving ACL homeomorphism $f : D \to \mathbb{C}$, the Jacobian $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2$ is nonnegative a.e., see [3], p. 10. In this case, the *complex dilatation* μ_f of f is the ratio $\mu(z) = f_{\overline{z}}/f_z$, if $f_z \neq 0$ and $\mu(z) = 0$ otherwise , and the *dilatation* K_f of f is defined as $K_\mu(z)$, see (1.2). Note that $|\mu(z)| \leq 1$ a.e. and $K_\mu(z) \geq 1$ a.e.

The *tangential dilatation* of (1.1) with respect to $z_0 \in D$ is

$$K^T_\mu(z, z_0) \;=\; \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2} \; . \tag{1.3}$$

Reasons for the name is explained in Section 3.

Conditions for the existence and uniqueness of ACL homeomorphic solutions for the Beltrami equation can be given in terms of the maximal dilatation $K_\mu(z)$. In particular, it was proved that, if $K_\mu(z)$ has a BMO majorant, then the Beltrami equation (1.1) has a homeomorphic ACL solution, see e.g. [4]. Many conditions of existence for the Beltrami equation have been formulated in terms of integral and measure constraints on K_μ, see e.g. [5 - 14]. The existence criteria established in the present paper are expressed in terms of the tangential dilatations $K^T_\mu(z, z_0)$ as in [15] and [16].

Below a point $z \in D$ is called a *regular point* for a mapping $f : D \to \mathbb{C}$ if f is differentiable at z and $J_f(z) \neq 0$.

2 Finite mean oscillation

Recall that a real valued function $\varphi \in \mathrm{L}^1_{loc}(D)$ is said to be of *bounded mean oscillation* in D, abbr. $\varphi \in$ BMO(D) or simply $\varphi \in$ BMO, if

$$\|\varphi\|_* = \sup_{B \subset D} \;\; -\!\!\!\!\!\!\int_B |\varphi(z) - \varphi_B| \; dxdy \;\; < \;\; \infty \tag{2.1}$$

where the supremum is taken over all disks B in D and

$$\varphi_B = \fint_B \varphi(z)\ dxdy = \frac{1}{|B|}\int_B \varphi(z)\,dxdy \tag{2.2}$$

is the mean value of the function φ over B. It is well–known that $\mathrm{L}^\infty(\mathrm{D}) \subset \mathrm{BMO(D)} \subset \mathrm{L}^p_{loc}(\mathrm{D})$ for all $1 \le p < \infty$, see e.g. [17]. A function φ in BMO is said to have *vanishing mean oscillation*, abbr. $\varphi \in$ VMO, if the supremum in (2.1) taken over all disks B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. The BMO space was introduced by John and Nirenberg, see [18], and soon became one of the main concepts in harmonic analysis, complex analysis and partial differential equations. BMO functions are related in many ways to quasiconformal and quasiregular mappings, see e.g. [19–24], as well as to the modern classes of mappings with finite distortion, see e.g. [7] and [25]. VMO has been introduced by Sarason, see [26].

Now, let D be a domain in the complex plane $\mathbb{C}$. We say that a function $\varphi : D \to \mathbb{R}$ has *finite mean oscillation* at a point $z_0 \in D$ if

$$d_\varphi(z_0) = \overline{\lim_{\varepsilon\to 0}} \quad \fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)|\ dxdy \ < \ \infty \tag{2.3}$$

where

$$\overline{\varphi}_\varepsilon(z_0) = \fint_{D(z_0,\varepsilon)} \varphi(z)\ dxdy \ < \ \infty \tag{2.4}$$

is the mean value of the function $\varphi(z)$ over the disk $D(z_0, \varepsilon)$ with small $\varepsilon > 0$. Thus, the notion includes the assumption that φ is integrable in some neighborhood of the point z_0. We call $d_\varphi(z_0)$ the *dispersion* of the function φ at the point z_0. We say that a function $\varphi : D \to \mathbb{R}$ is of *finite mean oscillation* in D, abbr. $\varphi \in \mathrm{FMO(D)}$ or simply $\varphi \in$ FMO, if φ has a finite dispersion at every point $z \in D$.

2.5. Remark. Note that, if a function $\varphi : D \to \mathbb{R}$ is integrable over $D(z_0, \varepsilon_0) \subset D$, then

$$\fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)|\ dxdy \ \le \ 2 \cdot \overline{\varphi}_\varepsilon(z_0) \tag{2.6}$$

and the right side in (2.6) is continuous in the parameter $\varepsilon \in (0, \varepsilon_0]$ by the absolute continuity of the indefinite integral. Thus, for every $\delta_0 \in (0, \varepsilon_0)$,

$$\sup_{\varepsilon\in[\delta_0,\varepsilon_0]} \quad \fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)|\ dxdy \ < \infty\ . \tag{2.7}$$

If (2.3) holds, then

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \ \fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \ dxdy \ < \infty \ . \tag{2.8}$$

The number in (2.8) is called the *maximal dispersion* of the function φ in the disk $D(z_0, \varepsilon_0)$.

2.9. Proposition. *If, for some collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \ dxdy \ < \infty \, , \tag{2.10}$$

then φ is of finite mean oscillation at z_0.

Proof. Indeed, by the triangle inequality,

$$\fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| \ dxdy \ \le$$

$$\le \ \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \ dxdy \ + \ |\varphi_\varepsilon - \overline{\varphi}_\varepsilon(z_0)| \ \le$$

$$\le \ 2 \cdot \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \ dxdy \, ..$$

2.11. Corollary. *If, for a point $z_0 \in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{D(z_0,\varepsilon)} |\varphi(z)| \ dxdy \ < \ \infty \ , \tag{2.12}$$

then φ has finite mean oscillation at z_0.

2.13. Remark. Clearly BMO $\subset$ FMO. The example given in the end of this section shows that the inclusion is proper. Note that the function $\varphi(z) = log\frac{1}{|z|}$ belongs to BMO in the unit disk Δ, see e.g. [17], p. 5, and hence also to FMO. However, $\overline{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that the condition (2.12) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0.

A point $z_0 \in D$ is called a *Lebesgue point* of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon\to 0} \ \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi(z_0)| \ dxdy \ = \ 0 \ . \tag{2.14}$$

It is known that, for every function $\varphi \in L^1(D)$, almost every point in D is a Lebesgue point.

2.15. Corollary. *Every function $\varphi : D \to \mathbb{R}$, which is locally integrable, has a finite mean oscillation at almost every point in D.*

Below we use the notations $D(r) = D(0,r) = \{z \in \mathbb{C} : |z| < r\}$ and

$$A(\varepsilon, \varepsilon_0) = \{z \in \mathbb{C} : \varepsilon < |z| < \varepsilon_0\} . \tag{2.16}$$

2.17. Lemma. *Let $D \subset \mathbb{C}$ be a domain such that $D(1/2) \subset D$, and let $\varphi : D \to \mathbb{R}$ be a nonnegative function. If φ is integrable in $D(1/2)$ and of FMO at 0, then*

$$\int\limits_{A(\varepsilon,1/2)} \frac{\varphi(z)\, dxdy}{\left(|z| \log_2 \frac{1}{|z|}\right)^2} \le \ C \cdot \log_2 log_2 \frac{1}{\varepsilon} \tag{2.18}$$

for $\varepsilon \in (0, 1/4)$, where

$$C = 4\pi\ [\varphi_0 + 6d_0]\ , \tag{2.19}$$

φ_0 is the mean value of φ over the disk $D(1/2)$ and and d_0 is the maximal dispersion of φ in $D(1/2)$.

Versions of this lemma have been first established for BMO functions and $n = 2$ in [4] and [27] and then for FMO functions in [28] and [29]. An n-dimensional version of the lemma for BMO functions was established in [30].

3 Ring Q–homeomorphisms

Recall that, given a family of paths Γ in $\overline{\mathbb{C}}$, a Borel function $\rho : \overline{\mathbb{C}} \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in adm\,\Gamma$, if

$$\int\limits_{\gamma} \rho(z)\, |dz|\ \ge\ 1 \tag{3.1}$$

for each $\gamma \in \Gamma$. The *modulus* of Γ is defined by

$$M(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_{\mathbb{C}} \rho^2(z)\ dxdy\ . \tag{3.2}$$

We say that a property P holds for *almost every (a.e.)* path γ in a family Γ if the subfamily of all paths in Γ for which P fails has modulus zero. In particular, almost all paths in $\mathbb{C}$ are rectifiable.

Basically, there are three definitions of quasiconformality: analytic, geometric and metric. They are equivalent with the same parameter of quasiconformality K. By the analytic definition, a homeomorphism $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$, is K–quasiconformal, abbr. K–qc, if f is ACL and

$$ess\,sup\; K_\mu(z) = K < \infty \tag{3.3}$$

where μ is the complex dilatation of f. According to the geometric definition, f is K–quasiconformal if

$$sup\,\frac{M(f\Gamma)}{M(\Gamma)} = K < \infty \tag{3.4}$$

where the supremum is taken over all path families Γ in D with modulus $M(\Gamma) \neq 0$. It was noted by Ahlfors and Gehring that the supremum in (3.4) can be taken over special families yielding the same bound K. In particular, by [36] one may restrict to families of paths connecting the boundary components of rings in D.

Given a measurable function $K : D \to [1, \infty]$, we say that a sense-preserving ACL homeomorphism $f : D \to \overline{\mathbb{C}}$ is *K(z)-quasiconformal*, abbr. $K(z)$-qc, if

$$K_f(z) \leq K(z) \quad \text{a.e.} \tag{3.5}$$

Given a measurable function $Q : D \to [1, \infty]$, we say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a *Q-homeomorphism* if

$$M(f\Gamma) \leq \int\limits_D Q(z) \cdot \rho^2(z)\; dxdy \tag{3.6}$$

holds for every path family Γ in D and each $\rho \in adm\,\Gamma$. This term was introduced in [31], see also [30] and [32], and the inequality was used in [27] and [4] as a basic tool in studying BMO-qc mappings.

The inequality

$$M(f\Gamma) \leq \int\limits_D K_\mu(z) \cdot \rho^2(z)\; dxdy \tag{3.7}$$

for $\mu = \mu_f$ was obtained in [3], p. 221, for quasiconformal mappings. Note that K_μ cannot be replaced by a smaller function in (3.7) unless one restricts either to special families Γ or to special $\rho \in \operatorname{adm}\Gamma$. In this section we improve (3.7) for special Γ and ρ and then use it for deriving new criteria for the existence of homeomorphic solutions of the Beltrami equation (1.1).

Given a domain D and two sets E and F in $\overline{\mathbb{C}}$, $\Gamma(E,F,D)$ denotes the family of all paths $\gamma : [a,b] \to \overline{\mathbb{C}}$ which join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. We set $\Gamma(E,F) = \Gamma(E,F,\overline{\mathbb{C}})$ if $D = \overline{\mathbb{C}}$. A *ring domain*, or shortly a *ring* in $\overline{\mathbb{C}}$ is a doubly connected domain R in $\overline{\mathbb{C}}$. Let R be a ring in $\overline{\mathbb{C}}$. If C_1 and C_2 are the connected components of $\overline{\mathbb{C}} \setminus R$, we write $R = R(C_1, C_2)$. The *capacity* of R can be defined by

$$cap\ R(C_1, C_2) \ = \ M(\Gamma(C_1, C_2, R)) \ , \tag{3.8}$$

see e.g. [33] and [34]. Note that

$$M(\Gamma(C_1, C_2, R)) \ = \ M(\Gamma(C_1, C_2)) \ , \tag{3.9}$$

see e.g. Theorem 11.3 in [35].

Motivated by the ring definition of quasiconformality in [36], we introduce the following notion that localizes and extends the notion of a Q–homeomorphism. Let D be a domain in $\mathbb{C}$, $z_0 \in D$, $r_0 \le \operatorname{dist}(z_0, \partial D)$ and $Q : D(z_0, r_0) \to [0, \infty]$ a measurable function in the disk

$$D(z_0, r_0) = \{z \in \mathbb{C} : |z - z_0| < r_0\} \ . \tag{3.10}$$

Set

$$A(r_1, r_2, z_0) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\} \ , \tag{3.11}$$

$$C(z_0, r_i) = \{z \in \mathbb{C} : |z - z_0| = r_i\}, \quad i = 1, 2. \tag{3.12}$$

We say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a *ring Q–homeomorphism* at the point z_0 if

$$cap\ R(fC_1, fC_2) \ \le \int\limits_A Q(z) \cdot \eta^2(|z - z_0|)\ dxdy \tag{3.13}$$

for every annulus $A = A(r_1, r_2, z_0)$, $0 < r_1 < r_2 < r_0$, and for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\ dr \ = \ 1 \ . \tag{3.14}$$

Note that every Q–homeomorphism $f : D \to \overline{\mathbb{C}}$ is a ring Q–homeomorphism at each point $z_0 \in D$. Later on, we give other conditions on f which force it to be a ring Q–homeomorphism. For this goal, in addition to the maximal dilatation of a mapping we use two other dilatations.

Let z be a regular point for a mapping $f : D \to \mathbb{C}$. Given $\omega \in \mathbb{C}$, $|\omega| = 1$, the *derivative in the direction* ω of the mapping f at the point z is

$$\partial_\omega f(z) \;=\; \lim_{t\to+0} \frac{f(z+t\cdot\omega) - f(z)}{t}\ . \tag{3.15}$$

The *radial direction* at a point $z \in D$ with respect to the center $z_0 \in \mathbb{C}$, $z_0 \neq z$, is

$$\omega_0 \;=\; \omega_0(z, z_0) \;=\; \frac{z - z_0}{|z - z_0|}\ . \tag{3.16}$$

The *tangential dilatation* of f at z with respect to z_0 is defined by

$$K^T(z, z_0, f) \;=\; \frac{|\partial_T^{z_0} f(z)|^2}{|J_f(z)|} \tag{3.17}$$

where $\partial_T^{z_0} f(z)$ is the derivative of f at z in the direction $\tau = i\omega_0$. Note that if z is a regular point of f and $|\mu(z)| < 1$, $\mu(z) = f_{\overline{z}}/f_z$, then

$$K^T(z, z_0, f) \;=\; K_\mu^T(z, z_0) \tag{3.18}$$

where $K_\mu^T(z, z_0)$ are defined by (1.3). Indeed, the equality (3.18) follow directly from the calculation

$$\partial_T f = \frac{1}{r}\left(\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial\vartheta} + \frac{\partial f}{\partial\overline{z}}\cdot\frac{\partial\overline{z}}{\partial\vartheta}\right) = i\cdot\left(\frac{z - z_0}{|z - z_0|}\cdot f_z - \frac{\overline{z - z_0}}{|z - z_0|}\cdot f_{\overline{z}}\right) \tag{3.19}$$

where $r = |z - z_0|$ and $\vartheta = \arg(z - z_0)$ because $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2$.

The *big radial dilatation* of f at z with respect to z_0 is defined by

$$K^R(z, z_0, f) = \frac{|J_f(z)|}{|\partial_R^{z_0} f(z)|^2} \tag{3.20}$$

where

$$|\partial_R^{z_0} f(z)| \;=\; \min_{\omega\in\mathbb{C}, |\omega|=1} \frac{|\partial_\omega f(z)|}{|\mathrm{Re}\,\omega\overline{\omega_0}|}\ . \tag{3.21}$$

Here $\mathrm{Re}\,\omega\overline{\omega_0}$ is the scalar product of vectors ω and ω_0. In the other words, $\mathrm{Re}\,\omega\overline{\omega_0}$ is the projection of the vector ω onto the radial direction ω_0. Obviously, there is a unit vector ω_* such that

$$|\partial_R^{z_0} f(z)| \;=\; \frac{|\partial_{\omega_*} f(z)|}{|\mathrm{Re}\,\omega_*\overline{\omega_0}|}\ . \tag{3.22}$$

It is clear that
$$|\partial_R^{z_0} f(z)| \;\geq\; \min_{\omega\in\mathbb{C},|\omega|=1} |\partial_\omega f(z)| \tag{3.23}$$
and hence
$$K^R(z,z_0,f) \;\leq\; K_\mu(z) \tag{3.24}$$
and the equality holds in (3.24) if and only if the minimum in the right hand side of (3.23) is realized for the radial direction $\omega=\omega_0$.

Note that $|\partial_R^{z_0} f(z)| \neq 0$ and $\partial_T^{z_0} f(z) \neq 0$ at every regular point $z\neq z_0$ of f, see e.g. 1.2.1 in [37]. In view of (3.17), (3.18) and (1.3), the following lemma shows that the big radial dilatation and the tangential dilatation coincide at every regular point.

3.25. Lemma. *Let $z\in D$ be a regular point of a mapping $f:D\to\mathbb{C}$ with the complex dilatation $\mu(z)=f_{\overline{z}}/f_z$ such that $|\mu(z)|<1$. Then*
$$K^R(z,z_0,f) \;=\; \frac{\left|1-\frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2}\;. \tag{3.26}$$

Proof. The derivative of f at the regular point z in the arbitrary direction $\omega=e^{i\alpha}$ is the quantity $\partial_\omega f(z) \;=\; f_z \;+ f_{\overline{z}}\cdot e^{-2i\alpha}$, see e.g. [3], p. 17 and 182. Consequently,
$$X \;:\; = \;\frac{|\partial_R^{z_0} f(z)|^2}{|f_z|^2} \;=\; \min_{\alpha\in[0,2\pi]} \frac{|\mu(z)+e^{2i\alpha}|^2}{\cos^2(\alpha-\vartheta)} \;=\; \min_{\beta\in[0,2\pi]} \frac{|\nu-e^{2i\beta}|^2}{\sin^2\beta} \;=$$
$$=\; \min_{\beta\in[0,2\pi]} \frac{1+|\nu|^2-2|\nu|\cos(\kappa-2\beta)}{\sin^2\beta} \;=$$
$$=\; \min_{t\in[-1,1]} \frac{1+|\nu|^2-2|\nu|\cdot[(1-2t^2)\cos\,\kappa \pm 2t(1-t^2)^{1/2}\sin\,\kappa]}{t^2}$$
where $t=\sin\beta$, $\beta=\alpha+\frac{\pi}{2}-\vartheta$, $\nu=\mu(z)e^{-2i\vartheta}$ and $\kappa=\arg\nu=\arg\mu-2\vartheta$. Hence $X \;=\; \min\limits_{\tau\in[1,\infty]} \varphi_\pm(\tau)$ where $\tau \;=\; 1/\sin^2\beta$, $\;\varphi_\pm(\tau) \;=\; b+a\tau\pm c(\tau-1)^{1/2}$, $\;a \;=\; 1+|\nu|^2-2|\nu|\cos\kappa$, $\;b \;=\; 4|\nu|\cos\kappa$, $\;c \;=\; 4|\nu|\sin\kappa$. Since $\varphi_\pm'(\tau) \;=\; a\pm(\tau-1)^{-1/2}c/2$ the minimum is realized for $\tau \;=\; 1+c^2/4a^2$ under $(\tau-1)^{1/2} \;=\; \mp c/2a$, correspondingly, where the signes are agreed. Thus,
$$X \;=\; b+\left(a+\frac{1}{4}\frac{c^2}{a}\right)-\frac{1}{2}\frac{c^2}{a} \;=\; \frac{(1-|\nu|^2)^2}{1+|\nu|^2-2|\nu|\cos\kappa}$$

that implies (3.26).

Next, we recall some general properties of homeomorphisms in the Sobolev class $W^{1,2}_{loc}$.

3.27. Proposition. *Let $f : D \to \mathbb{C}$ be a homeomorphism of the class $W^{1,2}_{loc}$. Then f is differentiable a.e. and satisfies Lusin's property (N). If, in addition, f^{-1} belongs to the class $W^{1,2}_{loc}$, then*

$$J_f(z) \neq 0 \quad \text{a.e.} \tag{3.28}$$

The statement follows from the well–known results for $W^{1,2}_{loc}$ homeomorphisms, see e.g. [3], p. 121, 128–130 and 150, and the equivalence of the (N^{-1})-property and the property (3.28) for mappings f which are differentiable a.e., see Theorem 1 in [38]. Recall that a mapping $f : X \to Y$ between measurable spaces (X, Σ, μ) and (X', Σ', μ') is said to have the *Lusin (N)–property* if $\mu'(f(S)) = 0$ whenever $\mu(S) = 0$. Similarly, f has the *(N^{-1})–property* if $\mu(S) = 0$ whenever $\mu'(f(S)) = 0$.

Some prototypes of the following theorem can be found in [15], [16] and [39]. In these theorems, both $|\mu|$ and $\arg \mu$ are incorporated in modulus estimations.

3.29. Theorem. *Let $f : D \to \mathbb{C}$ be a sense–preserving homeomorphism of the class $W^{1,2}_{loc}$ such that $f^{-1} \in W^{1,2}_{loc}$. Then at every point $z_0 \in D$ the mapping f is a ring Q-homeomorphism with $Q(z) = K^T_\mu(z, z_0)$ where $\mu(z) = \mu_f(z)$.*

Proof. Fix $z_0 \in D$, and r_1 and r_2 such that $0 < r_1 < r_2 < r_0 \leq \operatorname{dist}(z_0, \partial D)$ and let $C_1 = \{z \in \mathbb{C} : |z - z_0| = r_1\}$ and $C_2 = \{z \in \mathbb{C} : |z - z_0| = r_2\}$. Set $\Gamma = \Gamma(C_1, C_2, D)$ and denote by Γ_* the family of all rectifiable paths $\gamma_* \in f\Gamma$ for which f^{-1} is absolutely continuous on every closed subpath of γ_*. Then $M(f\Gamma) = M(\Gamma_*)$ by the Fuglede theorem, see [43] and [35], because $f^{-1} \in ACL^2$, see e.g. [1], p. 8.

Fix $\gamma_* \in \Gamma_*$. Set $\gamma = f^{-1} \circ \gamma_*$ and denote by s and s_* natural (length) parameters of γ and γ_*, correspondingly. Note that the correspondence $s_*(s)$ between the natural parameters of γ_* and γ is a strictly monotone function and we may assume that $s_*(s)$ is increasing. For $\gamma_* \in \Gamma_*$, the inverse function $s(s_*)$ has the (N)–property and $s_*(s)$ is differentiable a.e. as a monotone function. Thus, $\frac{ds_*}{ds} \neq 0$ a.e. on γ by [38]. Let s be such that $z = \gamma(s)$ is a regular point for f and suppose that γ is differentiable at s

with $\frac{ds_*}{ds} \neq 0$. Let $r = |z - z_0|$ and let ω be a unit tangential vector to the path γ at the point $z = \gamma(s)$. Then

$$\left|\frac{dr}{ds_*}\right| = \frac{\frac{dr}{ds}}{\frac{ds_*}{ds}} = \frac{|\mathrm{Re}\,\omega\overline{\omega_0}|}{|\partial_\omega f(z)|} \leq \frac{1}{|\partial_R^{z_0} f(z)|} \tag{3.30}$$

where $|\partial_R^{z_0} f(z)|$ is defined by (3.21).

Now, let $\eta : (r_1, r_2) \to [0, \infty]$ be an arbitrary measurable function such that

$$\int\limits_{r_1}^{r_2} \eta(r)\ dr = 1\ . \tag{3.31}$$

By the Lusin theorem, there is a Borel function $\eta_* : (r_1, r_2) \to [0, \infty]$ such that $\eta_*(r) = \eta(r)$ a.e., see e.g. 2.3.5 in [40] and [41], p. 69. Let

$$\rho(z) = \eta_*(|z - z_0|)$$

in the annulus $A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$ and $\rho(z) = 0$ outside of A. Set

$$\rho_*(w) = \{\rho/|\partial_R^{z_0} f|\} \circ f^{-1}(w)$$

if $z = f^{-1}(w)$ is a regular point of f, $\rho_*(w) = \infty$ at the rest points of $f(D)$ and $\rho_*(w) = 0$ outside $f(D)$. Then by (3.30) and (3.31), for $\gamma_* \in \Gamma_*$,

$$\int\limits_{\gamma_*} \rho_* ds_* \ \geq \int\limits_{\gamma_*} \eta(r) \left|\frac{dr}{ds_*}\right| ds_* \ \geq \int\limits_{\gamma_*} \eta(r)\ \frac{dr}{ds_*}\ ds_* \ = \int\limits_{r_1}^{r_2} \eta(r)\ dr \ = \ 1$$

because the function $z = \gamma(s(s_*))$ is absolutely continuous and hence so is $r = |z - z_0|$ as a function of the parameter s_*. Consequently, ρ_* is admissible for all $\gamma_* \in \Gamma_*$.

By Proposition 3.27 f and f^{-1} are regular a.e. and have the property (N). Thus, by change of variables, see e.g. Theorem 6.4 in [42], we have in view of Lemma 3.25 that

$$M(f\Gamma) \leq \int\limits_{f(A)} \rho_*(w)^2 dudv \ = \int\limits_{A} \rho(z)^2\, K_\mu^T(z, z_0)\, dxdy \ =$$

$$= \int\limits_{A} K_\mu^T(z, z_0) \cdot \eta^2(|z - z_0|)\ dxdy\ ,$$

i.e., f is a ring Q–homeomorphism with $Q(z) = K^T_\mu(z, z_0)$.

If f is a plane $W^{1,2}_{loc}$ homeomorphism with a locally integrable $K_f(z)$, then $f^{-1} \in W^{1,2}_{loc}$, see e.g. [44]. Hence we obtain the following consequences of Theorem 3.29 which will be quoted below.

3.32. Corollary. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{loc}$ and suppose that $K_f(z)$ is integrable in a disk $D(z_0, r_0) \subset D$ for some $z_0 \in D$ and $r_0 > 0$. Then f is a ring Q-homeomorphism at the point $z_0 \in D$ with $Q(z) = K^T_\mu(z, z_0)$ where $\mu(z) = \mu_f(z)$.*

3.33. Corollary. *Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{loc}$ with $K_\mu \in L^1_{loc}$. Then f is a ring Q-homeomorphism at every point $z_0 \in D$ with $Q(z) = K_\mu(z)$ where $\mu(z) = \mu_f(z)$.*

We close this section with a convergence theorem which plays an important role in our scheme for deriving the existence theorems of the Beltrami equation.

3.34. Theorem. *Let $f_n : D \to \overline{\mathbb{C}}$, $n = 1, 2, \ldots$, be a sequence of ring Q–homeomorphisms at a point $z_0 \in D$. If f_n converge locally uniformly to a homeomorphism $f : D \to \overline{\mathbb{C}}$, then f is also a ring Q–homeomorphism at the point z_0.*

Indeed, it follows from the uniform convergence of the rings $f_nR(C_1, C_2)$ to the ring $fR(C_1, C_2)$, see [36, 45, 46].

4 Distortion estimates

For points $z, \zeta \in \overline{\mathbb{C}}$, the *spherical (chord) distance* $s(z, \zeta)$ between z and ζ is given by

$$s(z,\zeta) \;=\; \frac{|z-\zeta|}{(1+|z|^2)^{\frac{1}{2}}(1+|\zeta|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty \neq \zeta \;, \tag{4.1}$$

$$s(z,\infty) \;=\; \frac{1}{(1+|z|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty \;.$$

Given a set $E \subset \mathbb{C}$, $\delta(E)$ denotes the *spherical diameter* of E, i.e.,

$$\delta(E) \;=\; \sup_{z_1, z_2 \in E} s(z_1, z_2) \;. \tag{4.2}$$

Given a number $\Delta \in (0,1)$, a domain $D \subset \mathbb{C}$, a point $z_0 \in D$, a number $r_0 \le \operatorname{dist}(z_0, \partial D)$, and a measurable function $Q : D(z_0, r_0) \to [0, \infty]$, let $\mathfrak{R}_Q^\Delta$ denote the class of all ring Q–homeomorphisms $f : D \to \overline{\mathbb{C}}$ at z_0 such that

$$\delta(\overline{\mathbb{C}} \setminus f(D)) \ge \Delta \ . \tag{4.3}$$

Next, we introduce the classes $\mathfrak{B}_Q^\Delta$ and $\mathfrak{F}_Q^\Delta$ of certain qc mappings. Let $\mathfrak{B}_Q^\Delta$ denote the class of all quasiconformal mappings $f : D \to \overline{\mathbb{C}}$ satisfying (4.3) such that

$$K_\mu^T(z, z_0) \ = \ \frac{\left|1 - \frac{\overline{z - z_0}}{z - z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2} \ \le \ Q(z) \quad \text{a.e. in} \quad D(z_0, r_0) \tag{4.4}$$

where $\mu = \mu_f$. Similarly, let $\mathfrak{F}_Q^\Delta$ denote the class of all quasiconformal mappings $f : D \to \overline{\mathbb{C}}$ satisfying (4.3) such that

$$K_\mu(z) \ = \ \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \ \le \ Q(z) \quad \text{a.e. in} \quad D(z_0, r_0) \ . \tag{4.5}$$

4.6. Remark. By Corollaries 3.32 and the relations (3.24) and (3.18)

$$\mathfrak{F}_Q^\Delta \ \subset \ \mathfrak{B}_Q^\Delta \ \subset \ \mathfrak{R}_Q^\Delta \ . \tag{4.7}$$

The following lemma is based on well known capacity estimates by Gehring.

4.8. Lemma. *Let $f \in \mathfrak{R}_Q^\Delta$ and let $\psi_\varepsilon : [0, \infty] \to [0, \infty]$, $0 < \varepsilon < \varepsilon_0 < r_0$, be a one parameter family of measurable functions such that*

$$0 \ < \ I(\varepsilon) \ = \ \int\limits_{\varepsilon}^{\varepsilon_0} \psi_\varepsilon(t) \ dt \ < \ \infty \ , \qquad \varepsilon \in (0, \varepsilon_0) \ . \tag{4.9}$$

Then

$$s(f(\zeta), f(z_0)) \ \le \ \frac{32}{\Delta} \cdot exp\left(-\frac{2\pi}{\omega(|\zeta - z_0|)}\right) \tag{4.10}$$

for all $\zeta \in D(z_0, \varepsilon_0)$ where

$$\omega(\varepsilon) \ = \ \frac{1}{I^2(\varepsilon)} \int\limits_{A(\varepsilon)} Q(z) \cdot \psi_\varepsilon^2(|z - z_0|) \ dxdy \ . \tag{4.11}$$

Proof. Let E denote the component of $\overline{\mathbb{C}} \setminus fA$ containing $f(z_0)$ and F the component containing ∞ where $A = \{z \in \mathbb{C} : |\zeta - z_0| < |z - z_0| < r_0\}$. By the known Gehring lemma

$$cap\ R(E,F) \ \geq\ cap\ R_T\left(\frac{1}{\delta(E)\delta(F)}\right) \tag{4.12}$$

where $\delta(E)$ and $\delta(F)$ denote the spherical diameters of the continua E and F, correspondingly, and $R_T(t)$ is the Teichm¨uller ring

$$R_T(t)\ =\ \overline{\mathbb{C}} \setminus ([-1,0] \cup [t,\infty])\ , \qquad t\ >\ 1\ , \tag{4.13}$$

see e.g. 7.37 in [46] or [47]. It is also known that

$$cap\ R_T(t) = \frac{2\pi}{\log \Phi(t)} \tag{4.14}$$

where the function Φ admits the good estimates:

$$t + 1 \leq \Phi(t) \leq 16 \cdot (t+1) < 32 \cdot t, \quad t > 1\ , \tag{4.15}$$

see e.g. [46], p. 225–226, and (7.19) and (7.22) in [47]. Hence the inequality (4.12) implies that

$$cap\ R(E,F)\ \geq\ \frac{2\pi}{\log \frac{32}{\delta(E)\delta(F)}}\ .. \tag{4.16}$$

Thus,

$$\delta(E)\ \leq \frac{32}{\delta(F)}\ \exp\left(-\frac{2\pi}{cap\ R(E,F)}\right)\ , \tag{4.17}$$

consequently,

$$s(f(\zeta), f(z_0))\ \leq\ \frac{32}{\Delta} \cdot \exp\left(-\frac{2\pi}{cap\ R(fC, fC_0)}\right) \tag{4.18}$$

where $C_0 = \{z \in \mathbb{C} : |z - z_0| = r_0\}$ and $C = \{z \in \mathbb{C} : |z - z_0| = |\zeta - z_0|\}$.

Finally, choosing $\eta(r) = \psi_\varepsilon(r)/I(\varepsilon)$, $r \in (\varepsilon, \varepsilon_0)$, in (3.13) we obtain (4.10) from (4.18).

4.19. Theorem. *Let $f \in \mathfrak{R}^{\Delta}_{Q}$, and let $\psi : [0,\infty] \to [0,\infty]$ be a measurable function such that*

$$0\ <\ \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\ dt\ <\ \infty\ , \quad \varepsilon \in (0, \varepsilon_0)\ . \tag{4.20}$$

Suppose that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z)\cdot\psi^2(|z-z_0|)\ \ dxdy\ \le\ C\cdot\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\ dt \tag{4.21}$$

for all $\varepsilon\in(0,\varepsilon_0)$. *Then*

$$s(f(\zeta),f(z_0))\ \le\ \frac{32}{\Delta}\cdot exp\left(-\frac{2\pi}{C}\cdot\int\limits_{|\zeta-z_0|}^{\varepsilon_0}\psi(t)\ dt\right) \tag{4.22}$$

whenever $\zeta\in D(z_0,\varepsilon_0)$.

In the following theorem the estimate of distortion is expressed in terms of maximal dispersion of Q, see (2.8).

4.23. Theorem. *Let* $f\in\mathfrak{R}_Q^{\Delta}$ *for* $\Delta>0$ *and* Q *with finite mean oscillation at* $z_0\in D$. *If* Q *is integrable over a disk* $D(z_0,\varepsilon_0)\subset D$, *then*

$$s(f(\zeta),f(z_0))\ \le\ \frac{32}{\Delta}\cdot\left(\log\frac{2\varepsilon_0}{|\zeta-z_0|}\right)^{-\beta_0} \tag{4.24}$$

for every point $\zeta\in D(z_0,\varepsilon_0/2)$ *where*

$$\beta_0\ =\ \frac{1}{2}\ [q_0+6d_0]^{-1}\ , \tag{4.25}$$

q_0 *is the mean and* d_0 *the maximal dispersion of* $Q(z)$ *in* $D(z_0,\varepsilon_0)$.

Proof. The mean and the dispersion of a function over disks are invariant under linear transformations $w=(z-z_0)/2\varepsilon_0$. Hence, (4.24) follows by Theorem 4.19 and Lemma 2.17.

Another consequence of Lemma 4.8 can be formulated in terms of the *logarithmic mean* of Q over an annulus $A(\varepsilon)=A(\varepsilon,\varepsilon_0,z_0)=\{z\in\mathbb{C}:\varepsilon<|z-z_0|<\varepsilon_0\}$ which is defined by

$$M_{log}^{Q}(\varepsilon)\ =\ -\!\!\!\!\!\!\int\limits_{\varepsilon}^{\varepsilon_0} q(t)\ d\log t\ \ :\ =\ \frac{1}{\log\varepsilon_0/\varepsilon}\int\limits_{\varepsilon}^{\varepsilon_0} q(t)\ \frac{dt}{t} \tag{4.26}$$

where $q(t)$ denotes the mean value of Q over the circle $|z-z_0|=t$. Choosing in the expression (4.11) $\psi_\varepsilon(t)=1/t$ for $0<\varepsilon<\varepsilon_0$, and setting $\varepsilon=|\zeta-z_0|$ we have the following statement.

4.27. Corollary. *Let $Q : D(z_0, r_0) \to [0, \infty]$, $r_0 \leq dist\,(z_0, \partial D)$, be a measurable function, $\varepsilon_0 \in (0, r_0)$ and $\Delta > 0$. If $f \in \mathfrak{R}_Q^{\Delta}$, then*

$$s(f(\zeta), f(z_0)) \ \leq \ \frac{32}{\Delta} \cdot \left(\frac{|\zeta - z_0|}{\varepsilon_0} \right)^{1/M_{log}^{Q}(|\zeta - z_0|)} \tag{4.28}$$

for all $\zeta \in D(z_0, \varepsilon_0)$.

5 Existence theorems

We say that an ACL homeomorphism $f : D \to \mathbb{C}$ is a *ring solution* of the Beltrami equation (1.1) with a complex coefficient μ if f satisfies (1.1) a.e., $f^{-1} \in W_{loc}^{1,2}$ and f is a ring Q–homeomorphism at every point $z_0 \in D$ with $Q_{z_0}(z) = K_\mu^T(z, z_0)$, see (1.3). We show that ring solutions exist for wide classes of the degenerate Beltrami equations. The condition $f^{-1} \in W_{loc}^{1,2}$ given in the definition of a ring solution implies that a.e. point z is a regular point for the mapping f, i.e., f is differentiable at z and $J_f(z) \neq 0$.

The following lemma and corollary serve as the main tool in obtaining many criteria of existence of ring solutions for the Beltrami equation. In Theorem 5.9 the existence of a ring solution is established when at every point $z_0 \in D$ the tangential dilatation $K_\mu^T(z, z_0)$ is assumed to be dominated by a function of finite mean oscillation at z_0 in the variable z. In Theorem 5.11 below the condition for existence is formulated in terms of the mean of the tangential dilatation over infinitesimal disks. Since the maximal dilatation dominates the tangential dilatation, these two results obviously imply similar existence theorems in terms of conditions on the maximal dilatation, Theorem 5.13 and Corollary 5.15 below. The results on existence of regular ACL homeomorphic solutions (without the ring condition) in these latter theorem and corollary were established earlier in [29] with different proofs. The criterion for the existence in the last theorem in this section is formulated in terms of the logarithmic mean.

5.1. Lemma. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L_{loc}^1$. Suppose that for every $z_0 \in D$ there exist $\varepsilon_0 \leq dist\,(z_0, \partial D)$ and an one-parameter family of measurable functions $\psi_{z_0,\varepsilon} : (0, \infty) \to (0, \infty)$, $\varepsilon \in (0, \varepsilon_0)$, such that*

$$0 \ < \ I_{z_0}(\varepsilon) \ : = \ \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t) \ dt \ < \ \infty \ , \tag{5.2}$$

and

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0)\cdot\psi_{z_0,\varepsilon}^2(|z-z_0|)\;dxdy \;=\; o(I_{z_0}^2(\varepsilon)) \tag{5.3}$$

as $\varepsilon\to 0$. *Then the Beltrami equation (1.1) has a ring solution* f_μ.

Proof. Fix z_1 and z_2 in D. For $n\in\mathbb{N}$, define $\mu_n : D\to\mathbb{C}$ by letting $\mu_n(z)=\mu(z)$ if $|\mu(z)|\le 1-1/n$ and 0 otherwise. Let $f_n : D\to\mathbb{C}$ be a homeomorphic ACL solution of (1.1), with μ_n instead of μ, which fixes z_1 and z_2. Such f_n exists by the well-known existence theorem in the nondegenerate case, see e.g. [48], p. 98, cf. [3], p. 185 and 194. By Theorem 3.29 and Lemma 4.8, in view of (5.3), the sequence f_n is equicontinuous and hence by the Arzela–Ascoli theorem, see e.g. [49], p. 267, and [50], p. 382, it has a subsequence, denoted again by f_n, which converges locally uniformly to some nonconstant mapping f in D. Then, by Theorem 3.1 and Corollary 5.12 in [27] on convergence, f is $K(z)$-qc with $K(z)=K_\mu(z)$, see (3.5), and f satisfies (1.1) a.e. Thus, f is a homeomorphic ACL solution of (1.1). Moreover, by Theorems 3.29 and 3.34 f is a ring Q–homeomorphism, see (3.13), with $Q(z)=K_\mu^T(z,z_0)$ at every point $z_0\in D$.

Since the locally uniform convergence $f_n\to f$ of the sequence f_n is equivalent to the continuous convergence, i.e., $f_n(z_n)\to f(z_0)$ if $z_n\to z_0$, see [49], p. 268, and since f is injective, it follows that $g_n=f_n^{-1}\to f^{-1}=g$ continuously, and hence locally uniformly. By a change of variables which is permitted because f_n and g_n are in $W_{loc}^{1,2}$ we obtain that for large n

$$\int\limits_B |\partial g_n|^2\;dudv \;=\; \int\limits_{g_n(B)} \frac{dxdy}{1-|\mu_n(z)|^2}\;\le\int\limits_{B^*} Q(z)\;dxdy\;<\;\infty \tag{5.4}$$

where B^* and B are relatively compact domains in D and in $f(D)$, respectively, such that $g(\bar{B})\subset B^*$. The relation (5.4) implies that the sequence g_n is bounded in $\mathrm{W}^{1,2}(B)$, and hence $f^{-1}\in\mathrm{W}_{loc}^{1,2}(f(D))$, see e.g. [37], p. 319.

5.5. Corollary. *Let* $\mu : D\to\mathbb{C}$ *be a measurable function with* $|\mu(z)|<1$ *a.e.,* $K_\mu\in L_{loc}^1$, *and let* $\psi : (0,\infty)\to(0,\infty)$ *be a measurable function such that for all* $0<t_1<t_2<\infty$

$$0\;<\;\int\limits_{t_1}^{t_2}\psi(t)\;dt<\;\infty\;,\qquad\int\limits_{0}^{t_2}\psi(t)\;dt\;=\;\infty\;. \tag{5.6}$$

Suppose that for every $z_0 \in D$ there is $\varepsilon_0 < dist\,(z_0, \partial D)$ such that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu(z) \cdot \psi^2(|z-z_0|)\; dxdy \;\leq\; O\left(\int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\; dt\right) \tag{5.7}$$

as $\varepsilon \to 0$. Then (1.1) has a ring solution.

Lemma 5.1 yields the following theorem by choosing

$$\psi_{z_0,\varepsilon}(t) \;=\; \frac{1}{t \log \frac{1}{t}} \quad , \tag{5.8}$$

see also Lemma 2.17.

5.9. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0} such that*

$$K^T_\mu(z, z_0) \;\leq\; Q_{z_0}(z) \quad \text{ a.e.} \tag{5.10}$$

for some function $Q_{z_0}(z)$ of finite mean oscillation at the point z_0 in the variable z. Then the Beltrami equation (1.1) has a ring solution.

The following theorem is a consequence of Theorem 5.9 and Corollary 2.11.

5.11. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that at every $z_0 \in D$*

$$\overline{\lim_{\varepsilon\to 0}} \;\; ⨍_{D(z_0,\varepsilon)} \frac{\left|1 - \overline{\frac{z-z_0}{z-z_0}}\mu(z)\right|^2}{1-|\mu(z)|^2}\; dxdy \;<\; \infty\,. \tag{5.12}$$

Then the Beltrami equation (1.1) has a ring solution f_μ.

The following theorem is an important particular case of Theorem 5.9.

5.13. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$K_\mu(z) \;=\; \frac{1+|\mu(z)|}{1-|\mu(z)|} \;\leq\; Q(z) \;\in\; FMO\,. \tag{5.14}$$

Then the Beltrami equation (1.1) has a ring solution.

Since every ring solution is an ACL homeomorphic solution and since every BMO function is in FMO, the theorem generalizes and strengthens earlier results in [4, 27] about the existence of ACL homeomorphic solutions of the Beltrami equation when the conditions involve majorants of bounded mean oscillation.

5.15. Corollary. *If*

$$\overline{\lim_{\varepsilon\to 0}} \;\; -\!\!\!\!\!\!\int_{D(z_0,\varepsilon)} \frac{1+|\mu(z)|}{1-|\mu(z)|}\, dxdy \;<\; \infty \tag{5.16}$$

at every $z_0 \in D$, then (1.1) has a ring solution.

Applying Lemma 5.1 with $\psi(t) = 1/t$, we have also the following statement which is formulated in terms of the logarithmic mean, see (4.26), of $K_\mu^T(z, z_0)$ over the annuli $A(\varepsilon) = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}$ for a fixed $\varepsilon_0 = \delta(z_0) \le \operatorname{dist}(z_0, \partial D)$.

5.17. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. If at every point $z_0 \in D$ the logarithmic mean of K_μ^T over $A(\varepsilon)$ does not converge to ∞ as $\varepsilon \to 0$, i.e.,*

$$\liminf_{\varepsilon\to 0}\; M_{log}^{K_\mu^T}(\varepsilon) \;<\; \infty\ , \tag{5.18}$$

then the Beltrami equation (1.1) has a ring solution.

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