HAIT Journal of Science and Engineering C, Volume 4, Issues 1-2, pp. 295-304

Approximate controllability of structured systems with bounded input operators

Benzion Shklyar

Holon Institute of Technology, 52 Golomb St., Holon 58102, Israel
email: shk_b@hit.ac.il
Received 24 January 2006, accepted 30 January 2006

Abstract

The paper deals with approximate controllability problem for a linear distributed structured control system consisting of two distributed control systems, connected in a series. Approximate controllability conditions are obtained. Applications to the approximate controllability for control sytems with delay are considered.

1 Introduction and problem statement

Research in the controllability theory has been very intensive in the last years, so it is absolutely impossible to describe the current state and recent progress within given paper. However the controllability theory has been developed, as a rule, for single control systems, whereas many technical applications use a number of control systems interrelated by different ways.

The goal of the present paper is to establish approximate controllability conditions for a control object containing two control systems interconnected into a series such that a control function from the first control system is an output of the second one. An output of an object under consideration is a state of the first system.

Let X, U, Z be Hilbert spaces, and let A, C be infinitesimal generators of strongly continuous C_0-semigroups $S_A(t)$ in X and $S_C(t)$ in Z correspondingly in the class $C_0[1, 2]$.

Consider the control evolution equation [1, 2] with scalar control[1]

[1] For the sake of the simplisity the research will be restricted by scalar control only.

$$\begin{aligned}\dot{x}(t) &= Ax(t)+bw(t), x(0)=x_0, &(1.1)\\ w(t) &= Ez(t),\ 0\le t<+\infty, &(1.2)\end{aligned}$$

where $z(t)$ is a mild solution of the another control equation of the form

$$\dot{z}(t)=Cz(t)+Du(t),\ z(0)=z_0,\ 0\le t<+\infty. \quad (1.3)$$

Here $x(t), x_0\in X,\ w(t)\in\mathbb{R},\ z(t), z_0\in Z,\ u(t)\in U,\ b\in X\ ,\ D:U\to Z$ is a linear bounded operator, $E:Z\to\mathbb{R}$ is a linear bounded onto operator.

Let $x(t,x_0,w(\cdot))$ be a mild solution of equation (1.1) with initial condition $x(0)=x_0$, generated by the control $w(t)$, and let $x(t,x_{0,}z_0,u(\cdot))$ be a mild solution of equation (1.1) with initial condition $x(0)=x_0$, generated by the control (1.2), where $z(t,z_0,u(\cdot))$ is a mild solution of equation (1.3) with initial condition $z(0)=z_0$.

Definition 1.1. *Equation (1.1) is said to be approximately controllable , if for each $x_1\in X$ and $\varepsilon>0$ there exist a time $t_1>0$ and a control $w(\cdot)\in L_2[0,t_1]$, , such that*

$$\| x_1-x(t_1,0,w(\cdot))\|<\varepsilon.$$

Definition 1.2. *Equation (1.1) is said to be approximately controllable by equation (1.3), if for each $x_1\in X$ and each $\varepsilon>0$ there exist a time $t_1>0$ and a control $u(\cdot)\in L_2([0,t_1],U)$, such that*

$$\| x_1-x(t_1,0,0.u(\cdot))\|<\varepsilon.$$

1.1 The assumptions

The assumptions on the operators A ,C, and D are listed below.

1. The operators A and C have purely point spectrums σ_A and σ_C with no finite limit points. Eigenvalues of both A and C have multiplicities 1. For the sake of simplicity we assume that $\sigma_A\cap\sigma_C=\oslash$.

2. All eigenvectors of the operators A and C are complete in X and Z respectively.

It is well-known [1-5], that if above assumptions hold, than: a mild solutions $x(t, x_0, w(\cdot))$ and $z(t, z_0, u(\cdot))$ of equations (1.1) and (1.3) are defined by the following representation formula

$$\begin{aligned} x(t,x_0,w(\cdot)) &= S_A(t)x_0 + \int_0^t S_A(t-\tau)bw(\tau)d\tau, && (1.4) \\ z(t,z_0,u(\cdot)) &= S_C(t)z_0 + \int_0^t S_C(t-\tau)Du(\tau)d\tau, && (1.5) \end{aligned}$$

where the integrals in (1.4)-(1.5) are understood in the Bochner sense [1].

Denote by

$$K_t = \left\{ \begin{array}{c} x \in X : \exists u(\cdot) \in L_2([0,t], U) : \\ x = \int_0^t S_A(t_1-\tau)\, bE\left(\int_0^\tau S_c(\tau-\theta)\, Du(\theta)d\theta\right) d\tau \end{array} \right\} \tag{1.6}$$

the attainable set of equation (1.1) at the time $t, t \geq 0$, generated by the control $u(\cdot) \in L_2([0,t], U)$.

Let $R_A(s) = (sI_A - A)^{-1}$, $R_C(s) = (sI_C - C)^{-1}$ be resolvent operators of the operators A and C, where I_A and I_C are unit operators in the spaces X and Z respectively.

2 Main results

Lemma 2.1. $K_{t_1} \subseteq K_{t_2}, \ \forall t_1 \leq t_2$.

Proof. We have

$$\int_0^t S_A(t_1-\tau)\, bE\left(\int_0^\tau S_c(\tau-\theta)\, Du(\theta)d\theta\right) d\tau = \int_0^t \Phi(t,\theta)\, Du(\theta)d\theta,$$

where $\Phi(t,\theta) = \int_\theta^t S_A(t_1-\tau)\, bES_c(\tau-\theta)\, d\tau$.

Denote $\xi = t_2 - t_1 + \theta$. One can write

$\int_0^{t_1} \Phi(t_1,\theta)\, Du(\theta)d\theta = \int_{t_2-t_1}^{t_2} \Phi(t_1, t_1-t_2+\xi)\, Du(t_1-t_2+\xi)d\xi =$

$= \int_{t_2-t_1}^{t_2} \left(\int_{t_1-t_2+\xi}^{t_1} S_A(t_1-\tau)\, bES_c((\tau+t_2-t_1)-\xi)\, d\tau\right) Du(t_1 - t_2 + \xi)d\xi.$

Now denote $\eta = \tau + t_2 - t_1$. We have

$= \int_{t_2-t_1}^{t_2} \left(\int_{t_1-t_2+\xi}^{t_1} S_A(t_1-\tau) \, bES_c((\tau+t_2-t_1)-\xi) \, d\tau \right) Du(t_1-t_2+\xi)d\xi =$

$= \int_{t_2-t_1}^{t_2} \left(\int_{\xi}^{t_2} S_A(t_2-\eta) \, bES_c(\eta-\xi) \, d\tau \right) Du(t_1-t_2+\xi)d\theta =$

$= \int_0^{t_2} \Phi(t_2,\xi) \, Du_2(\xi)d\xi$, where

$$u_2(\xi) = \begin{cases} 0, & \text{if } \ 0 \le \xi < t_2 - t_1, \\ u(t_1-t_2+\xi), & \text{if } \ \xi \ge t_2 - t_1. \end{cases}$$

Obviously, if $u(\cdot) \in L_2([0,t_1],U)$, then $u_2(\cdot) \in L_2([0,t_2],U)$. This proves the lemma.

Theorem 2.1. *For equation (1.1) to be approximately controllable on* $[0,t_1]$, $t_1 > T$, *by equation (1.2), it is necessary and sufficient, that*

1.
$$\operatorname{Ker}(\lambda I_A - A^*) \cap \operatorname{Ker} B^* = \{0\}, \forall \lambda \in \sigma_A; \tag{2.1}$$

2. $\exists \mu \in \sigma_C$, *such that*
$$\operatorname{Ker}(\mu I_C - C^*) \cap \operatorname{Ker} D^* = \{0\}, \tag{2.2}$$

and
$$\operatorname{Ker}(\mu I_C - C) \cap \operatorname{Ker} E = \{0\}. \tag{2.3}$$

Proof. **Sufficiency**. Assume that conditions (2.1)-(2.3) hold true.

Define the attainable set K generated by controls $w(\cdot)$, by the formula
$$K = \bigcup_{t \ge 0} K_t. \tag{2.4}$$

It follows from Lemma 2.1, that the set K is a linear subspace of the space X.

Let $g \in (K)^{\perp} = \{g \in X^* : (x,g) = 0, \forall x \in K\}$. By (2.4), we obtain
$$\left(\int_0^t S_A(t-\tau) bE \left(\int_0^{\tau} S_c(\tau-\theta) \, Du(\theta)d\theta \right) d\tau, g \right) \equiv 0, \tag{2.5}$$

$\forall u(\cdot) \in L_2([0,t],U), \forall t \ge 0$.

We have $\int_0^{\infty} S_A(t) \, e^{-st}dt = R_A(s)$, $\int_0^{\infty} S_C(t) \, e^{-st}dt = R_C(s)$[1, 2]. Applying to (2.5) the Laplace transform, we obtain by the Convolution Theorem [1, 2], that

$$(R_A(s)bER_C(s) \, Du, g) = (R_A(s)b, g) \, ER_C(s) \, Du \equiv 0, \ \forall s \notin \sigma_A \cup \sigma_C, \ \forall u \in U \tag{2.6}$$

for any complex $s \notin \sigma_A \cup \sigma_C$.

Afterward one can easily show that condition (2.6) is equivalent to the condition

$$(R_A(s)b, g) = 0 \tag{2.7}$$

or

$$ER_C(s)\,Du = 0, \tag{2.8}$$

$\forall s \notin \sigma_A \cup \sigma_C,\ \forall u \in U$.

Let the spectrum σ_A of the operator A consists of numbers $\lambda_j,\ j = 1, 2, ...$, enumerated in such a way that their absolute values are non-decreasing with respect to j (i.e. $|\lambda_j| \geq |\lambda_{j+1}|$).

Denote by $\varphi_j^1, \psi_j^1, j = 1, 2, ...$, eigenvectors of the operators A and A^* respectively.

Let the spectrum σ_C of the operator C consists of numbers $\mu_j,\ j = 1, 2, ...,$, enumerated in such a way that their absolute values are non-decreasing with respect to j (i.e. $\left|\mu_j\right| \geq \left|\mu_{j+1}\right|$).

Denote by $\varphi_j^2, \psi_j^2, j = 1, 2, ...$,eigenvectors of the operators A and A^* respectively.

One can easily show that conditions (2.2)–(2.3) are equivalent to the condition

$$(Du, \psi_j^2)E\varphi_j^2 \neq 0 \text{ for some } j \in \{1, 2, ..., .\} \tag{2.9}$$

It is well-known [1], that each $\mu_j \in \sigma_C, j = 1, 2, ...$ is a pole of the resolvent $R_C(s)$ and the function $ER_C(s)Du$ has the Laurent expansion

$$ER_C(s)Du = \gamma_j(s - \mu_j)^{-1} + R_{Cj}(s), j = 1, 2, ... \tag{2.10}$$

in a neighborhood of μ_j, where the operator-valued function $R_{Cj}(s),\ j = 1, 2, ...,$ is holomorphic in this neighborhood.

Applying to (2.10) the Caushy Theorem we obtain [6]

$$\gamma_j = (Du, \psi_j^2)E\varphi_j^2, j = 1, 2, ... \tag{2.11}$$

It follows from (2.8) and (2.10) that

$$\gamma_j = (Du, \psi_j^2)E\varphi_j^2 = 0, j = 1, 2, ..., \tag{2.12}$$

This contradicts to (2.9).

Therefore conditions (2.2)–(2.3) imply

$$ER_C(s_0)\,Du \neq 0 \text{ for some } s_0 \in \mathbb{C}. \tag{2.13}$$

By analiticity of $R_C(s)$ we have $ER_C\,(s)\,Du \neq 0$ in a neighborgood of s_0. By (2.6) we obtain that (2.7) holds in the same neighborhood. In virtue of analiticity of $R_A\,(s)$ we obtain that (2.7) holds for all regular s.

As well as for poles of $R_C\,(s)$ it is well-known that each $\lambda_j \in \sigma_A, j = 1, 2, ...$ is a pole of the resolvent $R_A(s)$ and the function $R_A(s)b$ has the Laurent expansion

$$R_A(s)b = \gamma_j(s - \lambda_j)^{-1} + R_{Aj}(s), j = 1, 2, ... \tag{2.14}$$

in a neighborhood of λ_j, where the operator-valued function $R_{Aj}(s)$, $j = 1, 2, ...,$ is holomorphic in this neighborhood, and [6]

$$\left(\gamma_{jl}, g\right) = \left(b, \psi_j^1\right)\left(\varphi_{j_j}^1, g\right), j = 1, 2, ... \tag{2.15}$$

It follows from (2.7) and (2.15), that

$$\left(\gamma_j, g\right) = \left(b, \psi_j^1\right)\left(\varphi_{j_j}^1, g\right) = 0, j = 1, 2, ... \tag{2.16}$$

One can also show that condition (2.1) is equivalent to the condition

$$\left(b, \psi_j^1\right) \neq 0, j = 1, 2, ... \tag{2.17}$$

It follows from (2.16) in (2.17), that

$$\left(\varphi_j^1, g\right) = 0, j = 1, 2, ..., . \tag{2.18}$$

By assumption 3 (see the list of assumptions) (2.18) implies $g = 0$.

This proves the sufficiency.

Necessity. Let equation (1.1) be approximately controllable by equation (1.3) (see Definition 1.2). If (2.1) does not hold, than equation (1.1) is not controllable [6], so it is not controllable by equation (1.3). This proves the necessity of condition (2.1).

If (2.2) or (2.3) do not hold, then

$$\begin{aligned} \text{either } \operatorname{Ker}\left(\mu I_C - C^*\right) \cap \operatorname{Ker} D^* &\neq \{0\} \\ \text{or } \operatorname{Ker}\left(\mu I_C - C\right) \cap \operatorname{Ker} E &\neq \{0\}, \forall \mu \in \sigma_C. \end{aligned} \tag{2.19}$$

One can easily show that condition (2.19) is equivalent to condition (2.12), so $ER_C\,(s)\,Du$ is an integer function for any $u \in U$. Since $ER_C\,(s)\,Du$ is obtained as a result of Laplace Transform for $ES_C\,(t)\,Du$, it has no singularity at infinity. Hence by Liouville Theorem we have $ER_C\,(s)\,Du = 0, \forall s \notin$

σ_C, $\forall u \in U$, i.e. (2.8) holds, so $g \in (K)^{\perp}$ for any $g \neq 0$ [2]. The last assertion contradicts to the approximate controllability of equation (1.1).

This proves the theorem.

3 Approximate controllability of linear differential control systems with delays

It is well-known that many classes of linear distributed control systems can be described by evolution equations of the form (1.1).

In this section we will show that a very important class of linear differential control systems completely fits into the framework of the previous section, so the controllability results of the previous section can be applied to establishing of the approximate controllability criterion of linear delay differential systems interconnected by the way of the previous section.

Consider a linear differential-difference systems of the form [7]

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1) + b_0 v(t), 0 < h_1, \tag{3.1}$$

$$x(0) = x^0,\ x(\tau) = \varphi(\tau) \text{ a.e. on } [-h_1, 0], \tag{3.2}$$

$$\begin{aligned} \dot{z}(t) &= C_0 z(t) + C_1 z(t - h_2) + Du(t), 0 < h_p, & (3.3)\\ z(0) &= z_0, z(\tau) = \xi(\tau) \text{ a.e. on } [-h_2, 0], & (3.4) \end{aligned}$$

where $v(t) = d^T\ z(t), d \in \mathbb{R}^r$ is a given constant vector.

Here
$x(t), x^0 \in \mathbb{R}^n, \varphi(\cdot) \in L_2([-h_1, 0], \mathbb{R}^n); v(t) \in \mathbb{R}$;
$A_j, j = 0, 1$, are constant $n \times n$ matrices, b_0 is a constant n-vector;
$z(t), z^0 \in \mathbb{R}^r, \psi(\cdot) \in L_2([-h_2, 0], \mathbb{R}^r), u(t), u^0 \in \mathbb{R}^l$;
$C_j, j = 0, 1$, are constant $r \times r$ matrices, D is a constant $r \times l$ matrix.
We consider the Hilbert spaces [8]

$$\begin{aligned} X &= \mathbb{R}^n \times L_2([-h_1, 0], \mathbb{R}^n) = M_2^n[-h_1, 0], & (3.5)\\ Z &= \mathbb{R}^r \times L_2([-h_2, 0], \mathbb{R}^r) = M_2^r[-h_2, 0] & (3.6) \end{aligned}$$

as state spaces of systems (3.1) and (3.3) respectively, and $V = \mathbb{R}^r$.

It is known that the problem (3.1)-(3.2) is well-posed [4, 5, 9], so it can be described by particular case of problem (1.1)-(1.2) where the state

[2] In this case equation (1.3) has no nontrivial output defined by (1.2), so $K = \{0\}$. Thus condition (2.3) is a criterion of the existence of nontrivial input generated by the operator E, for equation (1.2).

space X is defined by (3.5) (see, for example, [8-10]); one can show that the corresponding operator A satisfies the assumptions (see subsection "The Assumptions"). We assume $\det A_1 \neq 0$ to provide the assumption 2 of subsection "The Assumptions".

The problem (3.3)-(3.4) is well-posed also, so it can be described by particular case of problem (1.3)-(1.4) , where the state space Z is defined by (3.6) . One can show by the same way that the corresponding operator C satisfies the assumptions (see subsection "The Assumptions") also. Denote the identity $k \times k$ matrix by $I_k, k = n, r$.

Definition 3.1. *System (3.1) is said to be approximately controllable by hereditary equation (3.3), if for any* $\varepsilon > 0$ *and for any* $(x_1, x(\cdot)) \in M_2^n[-h_1, 0]$ *there exist* $(u_0, \xi(\cdot)) \in U$ *and time moment* $t_1 > 0$ *such that the corresponding solution* $x(t)$ *of system (3.1) satisfies the inequality*[3] $\| x_1 - x(t) \| < \varepsilon, t_1 - h_1 \leq t \leq t_1$.

Theorem 3.1. *System (3.1) is approximately controllable by system (3.3), if and only if:*

1.
$$\operatorname{rank}\left\{\lambda I_n - A_0 - A_1 e^{-\lambda h_1}, b\right\} = n, \ \forall \lambda \in \sigma_A. \tag{3.7}$$

$\exists \mu \in \sigma_C$, *such that*

2.
$$\operatorname{rank}\left\{\mu I_r - C_0 - C_1 e^{-\mu h_2}, D\right\} = r, \tag{3.8}$$

and

3.
$$\operatorname{rank}\left\{\mu I_r - C_0^T - C_k^T e^{-\mu h_2}, d\right\} = r. \tag{3.9}$$

Proof. Denote: $\chi(t) = (x(t), x_t), \upsilon(t) = (u(t), u_t)$, where $x_t = x(t+\theta), -h_1 \leq \theta \leq 0, u_t = u(t+\theta), -h_2 \leq \theta \leq 0$. Let $\mathcal{AC}([-h_1, 0], \mathbb{R}^n)$ and $\mathcal{AC}([-h_2, 0], \mathbb{R}^r)$ be spaces of absolutely continuous $\mathbb{R}^n$-valued functions on $[-h_{1m}, 0]$ and $\mathbb{R}^r$-valued functions on $[-h_2, 0]$ correspondingly.

It is well-known [8-10], that systems (3.1) and (3.3) can be written in the form (1.1)-(1.2) in state spaces X and U defined above, where the infinitesimal operator A of the C_0-semigroup $S_A(t)$ is defined in the domain

$$D(A) = \left\{ \begin{array}{c} \chi = \left(x^0, \varphi(\cdot)\right) \in X : \varphi(\cdot) \in \mathcal{AC}([-h_1, 0], \mathbb{R}^n), \\ \dot{\varphi}(0) = A_0 \varphi(0) + A_1 \varphi(-h_1) \end{array} \right\}$$

[3] The norm is considered in the space $M_2^n[-h_{1m}, 0]$.

by

$$A\chi = \left(A_0\varphi\left(0\right) + A_1\varphi\left(-h_1\right), \dot{\varphi}\left(\cdot\right)\right), \tag{3.10}$$

and infinitesimal operator C of the C_0-semigroup $S_C\left(t\right)$ is defined in the domain

$$D\left(C\right) = \left\{ \begin{array}{c} z = \left(u^0, \psi\left(\cdot\right)\right) \in Z : \psi\left(\cdot\right) \in AC\left(\left[-h_2, 0\right], \mathbb{R}^r\right), \\ \dot{\psi}\left(0\right) = C_0\psi\left(0\right) + C_1\psi\left(-h_2\right) \end{array} \right\}$$

by

$$Cu = \left(C_0\psi\left(0\right) + C_1\psi\left(-h_2\right), \dot{\psi}\left(\cdot\right)\right), \tag{3.11}$$

the input operator and operator $B : V \to X$ is given by the formula

$$Bv = \left(B_0 v_0, 0\right), \forall v_0 \in \mathbb{R}^r, \tag{3.12}$$

where the operator $E : U \mapsto \mathbb{R}^r$ is defined by $Ez = d^T z^0, \forall z = \left(z^0, z\left(\cdot\right)\right) \in Z$.

Here all the assumptions 1-4 of the subsection "The assumptions" for above operators A and C are valid with $T = \max\left\{nh_{1,}rh_2\right\}$ [6, 7, 11, 12]. It has been proved [12] that condition (3.7) for system (3.1) is equivalent to condition (2.1) for equation (1.1), and condition (3.8) is equivalent to condition (2.2) for equation (1.1), and condition (3.9) is equivalent to condition (2.3) for equation (1.1).

The proof of the theorem is completed by Theorem 2.1.

References

[1] E. Hille and R. Philips, *Functional Analysis and Semi-Groups* (AMS, 1957).

[2] M. Krein, *Linear Differential Equations in Banach Spaces* (Moscow, Nauka, 1967).

[3] A. Lasieska, *Unified theory for abstract parabolic boundary problems - a semigroup approach*, Appl. Math. Optimiz. **10**, 225-286 (1983).

[4] D. Salamon, *Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach*, Trans. Amer. Math. Soc. **300**, 383-431 (1987).

[5] G. Weiss, *Admissibility of unbounded control operators*, SIAM J. Contr. and Optimiz. **27**, 527-545 (1989).

[6] B. Shklyar, *On attainable set and controllability for abstract control problem with unbounded input operator*, Proceeding of the Institute of Mathematics, NAS of Belarus **7**, 142-152 (2001).

[7] R. Bellmann and K. Cooke, *Differential-Difference Equations* (New York Academic Press, London, 1963).

[8] A. Manitius and R. Triggiani, *Function space controllability of linear retarded systems: a derivation from abstract operator conditions*, SIAM J. Control and Optimiz. **6**, 599-643 (1978).

[9] J. Hale, *Theory of Functional Differential Equations* (Springer, N.Y.-Heidelberg-Berlin, 1977).

[10] S. Shimanov, *To the theory of linear differential equations with aftereffect*, Differential Equations **1**, 102-116 (in Russian, 1965).

[11] H. Banks, M. Jakobs, and C. Langenhop, *Characterization of the controlled states in* $W_2^{(1)}$ *of linear hereditary systems*, SIAM Journal of Control **13**, 611-649 (1975).

[12] B. Shklyar, *On the approximate controllability of distribute-parameter systems,* Differential Equations **27**, 326-334 (1991).