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One-dimensional disordered sample as a quantum potential well: localization and resonant transmission *

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Abstract

Resonant transmission and localization of energy in $1D$ disordered systems have been studied. It is shown how the existence and properties of a resonance at a given frequency are related to the internal structure of the random realization. A mapping of the stochastic scattering problem onto a deterministic quantum problem is developed. It enables one to exploit quantum mechanical formulas for the quantitative description of the spectral density, transmission coefficients and spatial energy distributions at the resonances using the total length of the sample and the localization length as the fitting parameters. The validity of the analytical results derived from the mapping had been checked by extensive numerical simulations.

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There are several definitions and indications of the localization of wave fields in open disordered systems. The most direct and unambiguous one is the existence of the quasi-discrete set of frequencies (energies) at which the wave (quantum particle) is literally localized (trapped) in a small (compare to the total size) area inside the system [1, 2]. In this paper, we investigate the statistical properties of the localized states (modes, or resonances) in a one-dimensional ($1D$) media and their connection to the internal structure of disordered samples. It is shown that at each resonant frequency the field is localized in a segment that is in itself transparent for this frequency and is bounded from both sides by opaque ones, constituting an effective resonant cavity with high Q-factor. The stochastic scattering problem is mapped onto a quantum mechanical problem of tunnelling and resonant transmission through an effective two-humped regular potential. With the formulae derived on the basis of this mapping, we find statistical characteristics of the resonances, namely, their spectral density, spectral and spatial widths, and amplitudes. We show that the transmission coefficient at a resonant frequency, being independent of the total length, L, of the sample, is determined only by the location of the effective cavity, and therefore is totally different from the typical transmission that decays exponentially with increasing L. The maximal transmission, T, is provided by the modes that are located in the centre of the sample, while the highest energy concentration takes place in cavities shifted towards the input. The number of frequencies with $T \simeq 1$ is shown to be $L/l_{loc} \gg 1$ times smaller than the total number of resonances (l_{loc} is the localization length). The results enable us to solve an inverse problem, namely, to predict (with some probability) for each resonance the position of the effective cavity and the maximal pumping amplitude, using the total length of the sample and the localization length as the fitting parameters, and the transmission coefficient as the only (measurable) input datum. The analytical results deduced from the quantum-mechanical analogy are in close agreement with the results of the numerical experiments.

We consider a plane monochromatic wave with unit amplitude incident from the left ($x < 0$) on a $1D$ disordered sample consisted of N layers with random widths and fluctuating refractive indices. Typical frequency dependence of the transmission coefficient on the wavelength λ is shown in Fig. 1. One can see that along with a continuum of wavelengths for which the transmission coefficient is exponentially small ($\sim \exp(-2L/l_{loc})$) (Fig. 1, λ_a), there is a discrete set of points on λ-axis (like λ_b and λ_c in Fig. 1), where $T(\lambda)$ has well pronounced narrow maxima . The amplitude distributions induced by the corresponding waves inside the sample are shown in Fig. 2.

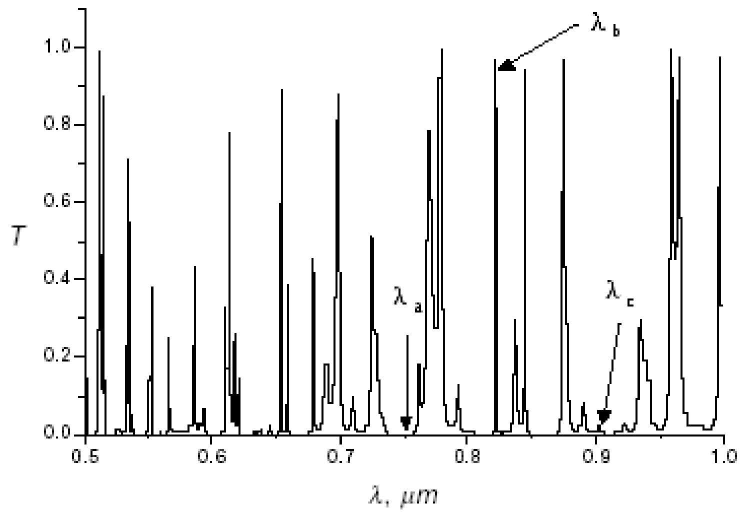

Figure 1: Transmission coefficient as a function of the wavelength.

While at a typical frequency (realization) the amplitude of the field decreases exponentially from the input (Fig. 2, λ_a), the resonances, (Fig. 2, λ_b, λ_c), exhibit essentially non-monotonous spatial distribution of the amplitude. Ii is important to note that the localization of energy takes place for all resonances, and not only for those with the transmission coefficient close to one (compare $T(\lambda_b)$ and $T(\lambda_c)$ in Fig. 1). The amplitude of a maximum depends on its location in space, which in its turn is uniquely determined by the internal structure of the realization. As it is shown below, this fact provides means for evaluating the resonant amplitude and the coordinate of the point where the resonant mode is localized if the total transmission coefficient is known.

Fig. 3 demonstrates the connection between the amplitude distribution and the transparency of different parts of the sample. It presents the intensities of the field generated by resonant frequencies (λ_b, λ_c in Fig. 1) inside the whole sample (thin curve), and in their left (thick gray curve), middle (thick black curve), and right (thick light gray curve) parts when they are taken as a separate sample each. It is seen that the middle sections where the energy is concentrated are almost transparent for the wave, while the side parts are practically opaque for the resonant frequency. Numerical analysis shows that this structure is inherent in all resonant realizations.

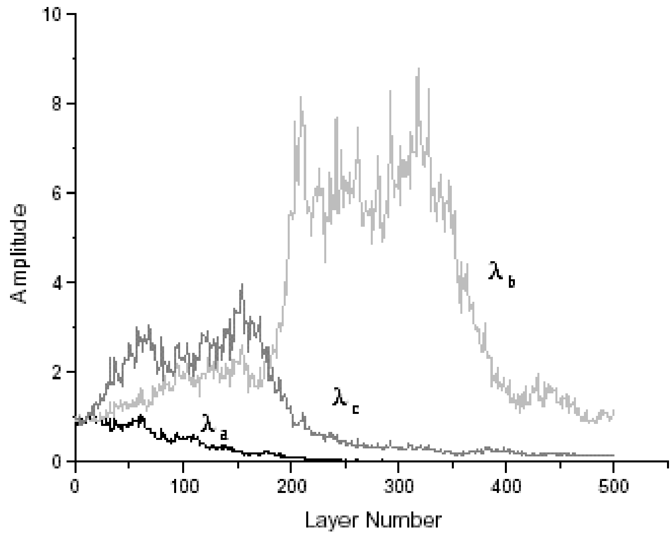

Figure 2: Amplitude of the field inside the sample as a function of the coordinate for three wavelengths marked in Fig. 1.

Structures of this type have been studied in the quantum mechanical problem of the tunnelling and resonant transmission of particles trough a regular (non-random) potential profile consisted of a well bounded by two potential barriers [3]. Although the physics of propagation in each system is totally different (interference of the multiply scattered random fields in a randomly layered medium, and tunnelling through a regular two-humped potential), the similarity turns out to be very close. Indeed, in both cases the transmission coefficients are exponentially small for most of frequencies (energies), and have well-pronounced resonant maxima (sometimes of order of unity) at discrete points corresponding to the eigen levels of each system. The energy at a resonant frequency is localized in the transparent part, and the total transmission depends drastically on the position of this part. More than that, even qualitatively the intensity distributions, Fig. 2, and the corresponding values of the transmission coefficients compare favorably with those calculated for a two-humpered potential profile, if its parameters have been chosen properly.

To map the stochastic scattering problem onto a deterministic quantum mechanical one, that is to say, to construct an appropriate effective regular potential that provides the same transmission and the interior intensity

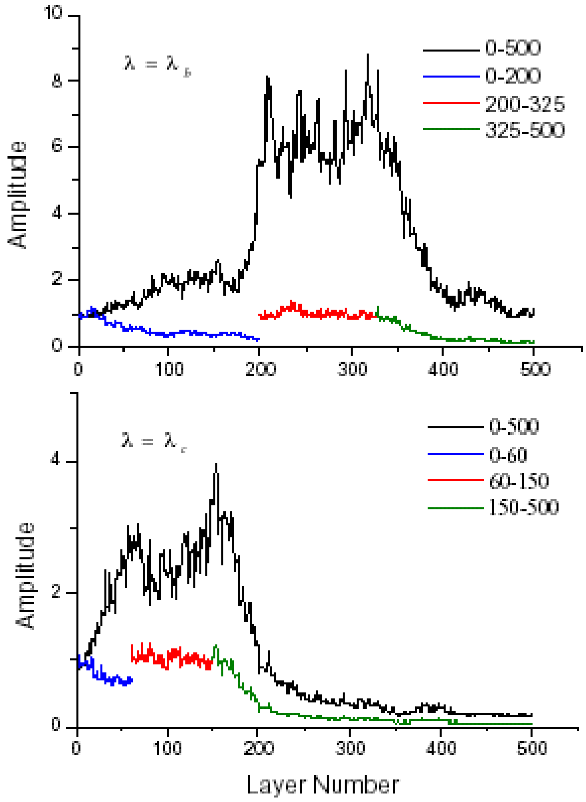

Figure 3: Amplitude of the field as a function of the coordinate inside the whole sample (black curves), in the left (blue curves), middle (red curves), and right (green curves) parts taken as a separate sample each. $\lambda = \lambda_b$ and $\lambda = \lambda_c$.

distribution, we refer to the theory of the tunnelling through two barriers separated by a potential well [3]. In WKB approximation all quantities of interest can be expressed via the length of the well and the tunnelling amplitudes (transmission coefficients) of the adjacent barriers. To determine properly the length of the (effective) well in disordered systems we note that the existence of a transparent segment inside a random sample is the result of a very specific (and therefore rare) combination of phase relations. Obviously, the longer such segment is, the less is the probability of its occurrence. On the other hand, the typical scale in the localized regime is the localization length. Hence, the minimal and, therefore, the most probable size of the effective well is l_{loc}. If the centre of the transparent segment of a resonant realization is shifted on the distance d from the centre of the sample, the lengths of non-transparent parts are

$$l_{1,2} = \frac{L - l_{loc}}{2} \pm d. \tag{1}$$

These parts are typical random realizations with the exponentially small transmission coefficients

$$T_{1,2} = \exp\left(-2l_{1,2}/l_{loc}\right). \tag{2}$$

Substitution of Eqs. (1) and (2) into corresponding quantum-mechanical formulae [4] yields for the resonant transmission coefficients, T_{res}, and for the amplitudes of resonances, A_{res}:

$$T_{res}(d) = \frac{4}{[\exp(2d/l_{loc}) + \exp(-2d/l_{loc})]^2}\ , \tag{3}$$

$$|A_{res}(d)|^2 = \frac{8\exp(L/l_{loc} - 1 - 2d/l_{loc})}{[\exp(2d/l_{loc}) + \exp(-2d/l_{loc})]^2}\ . \tag{4}$$

Eq. (3) shows that the resonant transmission coefficient does not depend on the total length of the sample and is governed only by the position of the area of localization, in contrast to the typical transmission that decays exponentially with L increasing. Note that resonances with high transmission $T \simeq 1$ arise in the middle of the sample, $d \simeq 0$.

If $T_{res} \simeq 1$, then $d \simeq 0$, and Eq. (4) gives

$$|A_{res}| \simeq \sqrt{2}\exp\left(L/2l_{loc}\right) \gg 1\ . \tag{5}$$

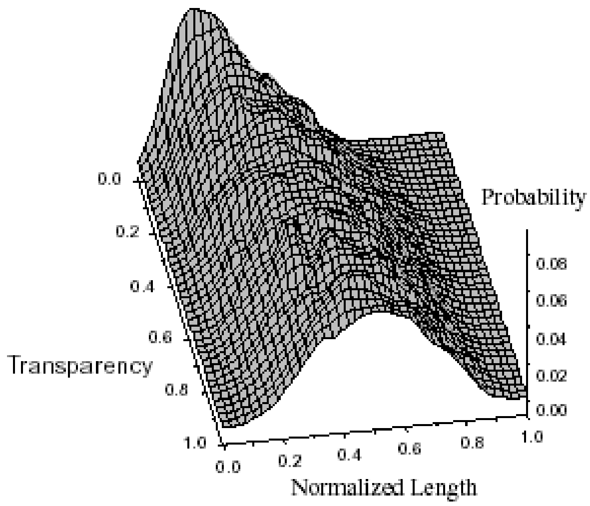

Figure 4: Probability for a mode localized at a given point to provide the value of the transmission coefficient T , as a function of the dimensionless coordinate x/L.

It can be shown from Eq. (4) that the maximum of the amplitude is reached when the transparent part is shifted from the centre of the sample towards the input at the distance

$$d_{\max} = -\frac{1}{4} \ln \frac{1}{3} \; l_{loc} \simeq -0.27 l_{loc} \; . \tag{6}$$

This shift is independent of the length of the sample.

From the corresponding quantum-mechanical formulae [4] it also follows that the distance (in k-space) between the resonances (localized modes) is

$$\Delta k \sim \frac{1}{L} \; , \tag{7}$$

while the typical interval between the resonantly transparent ($T \simeq 1$) modes is equal to

$$\Delta k_{tr} \sim \frac{1}{l_{loc}} \; . \tag{8}$$

It is easy to see, that $\Delta k / \Delta k_{res} \sim l_{loc}/L \ll 1$, which means that only a small fraction of resonances provides high transmission through the system.

This is physically clear and follows from the fact that a resonance with high transmission occurs when the transparent segment is located in the middle of the sample (to within the accuracy of l_{loc}), while the resonances with smaller transmission are observed in any non-symmetrical structure with arbitrary situated transparent segment. Assuming that the transparent segment can be found at any point of the sample with equal probability, we infer that resonances with $T \simeq 1$ are encountered $L/l_{loc} << 1$ times more rarely than all other ones. It also can be shown [4] that the typical width of resonances is

$$\delta k \sim \frac{1}{l_{loc}} \exp\left(-\frac{L}{l_{loc}}\right) , \tag{9}$$

that is to say, it decreases exponentially with the length of the sample. Note that in the localized regime $L \gg l_{loc} \gg \lambda$. This inequality justifies the validity of WKB approximation.

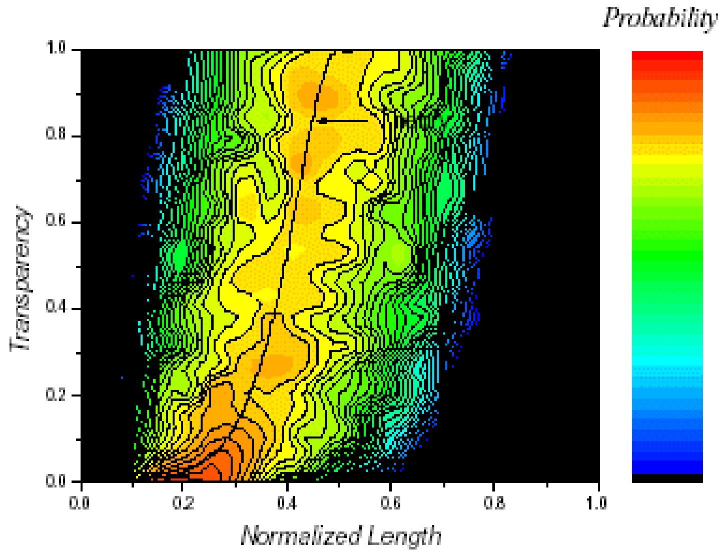

Figure 5: The same probability as in Fig. 4 presented by colors. Black line displays the transmission coefficient as a function of the coordinate of the point of localization calculated by Eq. (3).

To test the validity of the above-introduced analytical results, Eqs. (2) – (9), the numerical calculations of the spectrum of resonances, transmission coefficients, and spatial intensity distributions at resonant frequencies have been performed for more than 10^4 resonances. In the calculations we have considered samples with maximal number of layers $N = 1000$, and it was assumed that the refractive indices and sizes of the layers are independent random variables uniformly distributed in the ranges $n = 1 \pm 0.25$ and $d = 0.15 \pm 0.05$ μm respectively. The wavelength varied in the interval $0.5\mu m \leq \lambda \leq 1.5\mu m$. The localization length has been computed through the transmission coefficient as $l_{loc} = -2L < \ln T >^{-1}$. Note that, being a self-averaging quantity [5], l_{loc}^{-1} slightly fluctuates from sample to sample, and can be estimated from the transmission coefficient at a typical realization.

Depicted in Fig. 4 is the probability that a mode localized at a given point provides transmission coefficient T (dimensionless coordinate x/L is used). As is seen from the picture, the probability of high transitivity ($T \simeq 1$) has maximum for modes that are localized in the center of the random sample. As the location of an effective cavity shifts from the center, the corresponding transmission coefficient decreases, in agreement with Eq. (3). To make the comparison with the analytical results more convenient, the same (as in Fig. 4) probability density distribution is presented in Fig. 5 as a two-dimensional picture where different grayscale colors correspond to different probabilities. Black line displays the transmission coefficient at eigen frequencies calculated by Eq. (3) as a function of the coordinate of the corresponding point of localization. Note that shifted to the exit peaks of the field are not shown in Fig. 5.

The L-dependence of the eigen mode spacing and average half-width of the resonances retrieved from the numerical experiments fit well Eqs. (7), (8), (9). Average normalized field amplitude of localized modes (pumping rate of an effective resonant cavity) numerically calculated as a function of the coordinate of the cavity is well consistent with Eq. 4, and it shows that the effective cavities providing highest pumping rate are shifted from the center towards the input in accordance with Eq. (6). Interestingly enough, the seemingly rough analogy based on only one fitting parameter (localization length) performs surprisingly well. Indeed, not only the shift is independent on the total length and proportional to l_{loc} as predicted by Eq. (6), but also the coefficient in Eq. (6) coincides with that obtained in numerical simulations to within the accuracy 10%. We have also verified that the maximal peak amplitudes exponentially increase with the length of a sample and by the order of magnitude agree with the value given by Eq. (5).

To conclude, the resonant transmission through a disordered $1D$ system occurs due to the existence of a transparent (for a given resonant frequency) segment inside the system with the size of order of the localization length.

The mode structure of the sample, the transmission coefficient, and the intensity distribution at resonant frequencies depend on the positions of the segment. These dependencies are very robust, insensitive to the fine structure of the system, and therefore can be described by corresponding formulas for an effective regular potential profile. The fitting parameters are the total length and the localization length, which is a self-averaging quantity, it can be found, for example, from the transmission coefficient at typical (non-resonant) realizations. This feature enables to estimate the position of a resonant cavity by measuring the transmission coefficient at a general and at the corresponding resonant frequencies. From the first data the localization length can be obtained, the second one, T_{res}, is used to find the asymmetry parameter d (i.e. the coordinate of the effective cavity) from Eq. (3). Then Eq. (4) gives estimation for the pumped intensity. Therefore, although locally for $1D$ photons there is no analog to the quantum mechanical tunnelling (the effective energy of $1D$ photons is always higher than the corresponding potential barrier), macroscopically (on scales larger than the localization length) the problem of the propagation of light through a disordered $1D$ system can be reformulated in terms of an effective potential profile.

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